3.9 Orbits in General Relativity

Note. We now use some approximations to show the precession of the orbit of Mercury, mentioned in Section 3.3 and explicitly shown in Einstein's 1916 paper on general relativity. This result provided the first empirical evidence in support of the theory. Since this involves geodesics around a massive object at the (spatial) origin, me will be considering geodesic under the "Schwarzschild metric" of the previous section.

Note. We start with the Schwarzschild metric

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\, d\varphi^2 - r^2 \sin^2\varphi\, d\theta^2$$

and describe the path of a planet by a timelike geodesic

$$(x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau))$$

where

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

for $\lambda = 0, 1, 2, 3$. As in the previous section, we take $x^0 = t$, $x^1 = r$, $x^2 = \varphi$, and $x^3 = \theta$. The resulting Christoffel symbols are given in equation (153), page 214.

Note. With $\lambda = 2$ we have from the geodesic condition

$$\frac{d^2x^2}{d\tau^2} + \Gamma^2_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

and since the only nonzero Γ's with a superscript of 2 are Γ^2_{12}, Γ^2_{21}, and Γ^2_{33} we have

$$\frac{d^2x^2}{d\tau^2} + \Gamma^2_{12}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} + \Gamma^2_{21}\frac{dx^2}{d\tau}\frac{dx^1}{d\tau} + \Gamma^2_{33}\frac{dx^3}{d\tau}\frac{dx^3}{d\tau} = 0$$

or

$$\frac{d^2\varphi}{d\tau^2} + 2\left(\frac{1}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}\right) + (-\sin\varphi\cos\varphi)\left(\frac{d\theta}{d\tau}\right)^2 = 0.$$

We orient our axes such that when $\tau = 0$, we have $\varphi = \pi/2$ and $d\varphi/d\tau = 0$. So the planet starts in the plane $\varphi = \pi/2$ and due to symmetry remains in this plane. So we henceforth take $\varphi = \pi/2$. Now with $\lambda = 0$:

$$\frac{d^2x^0}{d\tau^2} + \Gamma^0_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

and since the only nonzero Γ's with a superscript of 0 are Γ^0_{10} and Γ^0_{01}, we have

$$\frac{d^2x^0}{d\tau^2} + \Gamma^0_{10}\frac{dx^1}{d\tau}\frac{dx^0}{d\tau} + \Gamma^0_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} = 0$$

or

$$\frac{d^2t}{d\tau^2} + 2\left(m'\frac{dr}{d\tau}\frac{dt}{d\tau}\right) = 0. \qquad (159a)$$

With $\lambda = 1$:

$$\frac{d^2x^1}{d\tau^2} + \Gamma^1_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

and since the only nonzero Γ's with a superscript of 1 are Γ^1_{00}, Γ^1_{11}, Γ^1_{22}, and Γ^1_{33}, we have

$$\frac{d^2x^1}{d\tau^2} + \Gamma^1_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \Gamma^1_{11}\frac{dx^1}{d\tau}\frac{dx^1}{d\tau} + \Gamma^1_{22}\frac{dx^2}{d\tau}\frac{dx^2}{d\tau} + \Gamma^1_{33}\frac{dx^3}{d\tau}\frac{dx^3}{d\tau} = 0$$

or

$$\frac{d^2r}{d\tau^2} + m'e^{2m-2n}\left(\frac{dt}{d\tau}\right)^2 + n'\left(\frac{dr}{d\tau}\right)^2$$
$$+(-re^{-2n})\left(\frac{d\varphi}{d\tau}\right)^2 + (-re^{-2n}\sin^2\varphi)\left(\frac{d\theta}{d\tau}\right)^2 = 0. \qquad (159b)$$

(since we have $\varphi = \pi/2$, $\sin^2\varphi \equiv 1$). With $\lambda = 3$

$$\frac{d^2x^3}{d\tau^2} + \Gamma^3_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

and since the only nonzero Γ's with a superscript of 3 are Γ^3_{13}, Γ^3_{31}, Γ^3_{23}, and Γ^3_{32}, we have

$$\frac{d^2x^3}{d\tau^2} + \Gamma^3_{13}\frac{dx^1}{d\tau}\frac{dx^3}{d\tau} + \Gamma^3_{31}\frac{dx^3}{d\tau}\frac{dx^1}{d\tau} + \Gamma^3_{23}\frac{dx^2}{d\tau}\frac{dx^3}{d\tau} + \Gamma^3_{32}\frac{dx^3}{d\tau}\frac{dx^2}{d\tau} = 0$$

or

$$\frac{d^2\theta}{d\tau^2} + 2\left(\frac{1}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}\right) + 2\left(\cot\varphi\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau}\right) = 0$$

or since $\varphi = \pi/2$

$$\frac{d^2\theta}{d\tau^2} + 2\left(\frac{1}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}\right) = 0. \qquad (159c)$$

Note. Now by the Chain Rule $m'\frac{dr}{d\tau} = \frac{dm}{dr}\frac{dr}{d\tau} = \frac{dm}{d\tau}$, and dividing (159a) by $dt/d\tau$ gives

$$\frac{d^2t/d\tau^2}{dt/d\tau} + 2\left(m'\frac{dr}{d\tau}\right) = 0$$

or

$$\frac{d}{d\tau}\left[\ln\frac{dt}{d\tau}\right] = -2\frac{dm}{d\tau}.$$

Integration yields

$$\ln\left(\frac{dt}{d\tau}\right) = -2m + \text{ constant}$$

or

$$\frac{dt}{d\tau} = be^{-2m} = \frac{b}{\gamma} \qquad (160)$$

where b is some positive constant and we define $\gamma = e^{2m}$.

Note. Equation (159c) can be integrated (see page 63 for the process) to yield

$$r^2\frac{d\theta}{d\tau} = h \qquad (161)$$

where h is a positive constant.

Note. From the Schwarzschild metric with $\varphi = \pi/2$, $d\varphi = 0$, and $\gamma = e^{2m} = 1 - 2M/r$ we have

$$d\tau^2 = \gamma dt^2 - \gamma^{-1} dr^2 - r^2 d\theta^2$$

or

$$\begin{aligned} 1 &= \gamma \left(\frac{dt}{d\tau}\right)^2 - \gamma^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 \\ &= \gamma \left(\frac{b}{\gamma}\right)^2 - \gamma^{-1} \left(\frac{dr}{d\theta}\frac{h}{r^2}\right)^2 - r^2 \left(\frac{h}{r^2}\right)^2 \qquad (162) \end{aligned}$$

by equation (160), the fact that

$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\frac{d\theta}{d\tau} = \frac{dr}{d\theta}\frac{h}{r^2}$$

(by the Chain Rule and equation (161)) and by equation (161). Multiplying (162) by γ yields

$$\gamma = b^2 - \left(\frac{dr}{d\theta}\frac{h}{r^2}\right)^2 - \gamma\frac{h^2}{r^2}$$

or since $\gamma = 1 - 2M/r$:

$$\left(1 - \frac{2M}{r}\right) = b^2 - \left(\frac{h}{r^2}\right)^2 \left(\frac{dr}{d\theta}\right)^2 - \left(1 - \frac{2M}{r}\right)\frac{h^2}{r^2}$$

or

$$\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{h^2}{r^2} = b^2 - 1 + \frac{2M}{r} + \frac{2M}{r}\frac{h^2}{r^2}.$$

Let $u = 1/r$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$ and the previous equation yields

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} = \frac{b^2-1}{h^2} + \frac{2M}{rh^2} + \frac{2M}{r^3}$$

or

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{b^2-1}{h^2} + \frac{2Mu}{h^2} + 2Mu^3.$$

Differentiation with respect to θ yields

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{2M}{h^2}\frac{du}{d\theta} + 6Mu^2\frac{du}{d\theta} \text{ (b and h are constants)}$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{M}{h^2} + 3Mu^2 \qquad (163)$$

where $u = 1/r$ and $h = r^2 d\theta/d\tau$ (constant).

Note. A similar analysis in the Newtonian setting yields

$$\frac{d^2u}{d\theta^2} + u = \frac{M}{h^2} \qquad (114)$$

(see page 193). So the only difference is the $3Mu^2$ term in (163) and we can think of this as the "relativistic term." Equation (114) has solution

$$u = \frac{M}{h^2}(1 + e\cos\theta).$$

Note. We can view (163) as a linear ODE (considering the left hand side) set equal to a nonhomogeneous term $\dfrac{M}{h^2} + 3Mu^2$. Now the term $3Mu^2$ is "small" as compared to M/h^2 (see page 226). We perturb this equation by replacing u with the approximate solution $M/h^2(1 + e\cos\theta)$ on the right hand side of (163) and consider

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= \frac{M}{h^2} + 3M\left(\frac{M}{h^2}(1 + e\cos\theta)\right)^2 \\ &= \frac{M}{h^2} + \frac{3M^3}{h^4}(1 + 2e\cos\theta + e^2\cos^2\theta).\end{aligned}$$

A solution to this (linear) ODE will then be an approximate solution to (163). The ODE is

$$\frac{d^2u}{d\theta^2} + u = \frac{M}{h^2} + \frac{3M^3}{h^4} + \frac{6M^3e}{h^4}\cos\theta + \frac{3M^3e^2}{2h^4} + \frac{3M^3e^2}{2h^4}\cos(2\theta). \quad (165)$$

(The last two terms follow from the fact that $\cos^2\theta = (1+\cos(2\theta))/2$.)

Lemma III-5. Let $A \in \mathbb{R}$. Then

1. $u = A$ is a solution of $d^2u/d\theta^2 + u = A$.

2. $u = (A/2)\theta\sin\theta$ is a solution of $d^2u/d\theta^2 + u = A\cos\theta$.

3. $u = (-A/3)\cos(2\theta)$ is a solution of $d^2u/d\theta^2 + u = A\cos(2\theta)$.

(The proof follows by simply differentiating.)

Note. Equation (165) is

$$\frac{d^2u}{d\theta^2} + u = \left(\frac{M}{h^2} + \frac{3M^3}{h^4} + \frac{3M^3e^2}{2h^4}\right) + \left(\frac{6M^3e}{h^4}\cos\theta\right) + \left(\frac{3M^3e^2}{2h^4}\cos(2\theta)\right)$$

and by Lemma III-5

$$\begin{aligned} u_p &= \left(\frac{M}{h^2} + \frac{3M^3}{h^4} + \frac{3M^3e^2}{2h^4}\right) + \left(\frac{3M^3e}{h^4}\theta\sin\theta\right) + \left(\frac{-M^3e^2}{2h^4}\cos(2\theta)\right) \\ &= \frac{M}{h^2}\left[1 + \frac{3M^2}{h^2}\left(1 + \frac{e^2}{2}\right) + \frac{3M^2e}{h^2}\theta\sin\theta - \frac{M^2e^2}{2h^2}\cos(2\theta)\right] \end{aligned}$$

is a (particular) solution to (165). Now the general solution to the homogeneous ODE

$$\frac{d^2u}{d\theta^2} + u = 0$$

is a linear combination of $u_h = \sin\theta$ and $u_h = \cos\theta$. Therefore, we can add any linear combination of these functions to the above particular solution to get another solution. We choose to add $\dfrac{M}{h^2}e\cos\theta$ (for reasons to be discussed shortly; we want to compare this solution to the Newtonian solution). We have

$$u = \frac{M}{h^2}\left[1 + \frac{3M^2}{h^2}\left(1 + \frac{e^2}{2}\right) + e\cos\theta + \frac{3M^2e}{h^2}\theta\sin\theta - \frac{M^2e^2}{2h^2}\cos(2\theta)\right]$$

Note. The term $\dfrac{3M^2}{h^2}\left(1+\dfrac{e^2}{2}\right)$ is small compared to 1 (8×10^{-8} for Mercury, see page 227). If we let

$$\alpha = 1+\frac{3M^2}{h^2}\left(1+\frac{e^2}{2}\right)$$

and define $e' = e/\alpha$ then

$$\frac{M\alpha}{h^2}(1+e'\cos\theta) = \frac{M}{h^2}(\alpha+e\cos\theta) = \frac{M}{h^2}\left(1+\frac{3M^2}{h^2}\left(1+\frac{e^2}{2}\right)+e\cos\theta\right).$$

Since $\alpha\approx 1$, $e'\approx e$ and we see that this part of the u function is approximately the same as the Newtonian solution. A similar argument shows that the "$\cos(2\theta)$" term causes little deviation from the Newtonian solution. Therefore we have that

$$u\approx\frac{M}{h^2}\left(1+e\cos\theta+\frac{3M^2e}{h^2}\theta\sin\theta\right).$$

Although the "$\theta\sin\theta$" may be very small initially, as θ increases "the term will have a *cumulative* effect over many revolutions" (see page 228). This effect is the observed perihelial advance.

Note. Since M^2/h^2 is small ($\approx 10^{-8}$ for Mercury, see page 228) we approximate

$$\cos\left(\frac{3M^2\theta}{h^2}\right)\approx 1,\qquad \sin\left(\frac{3M^2\theta}{h^2}\right)\approx\frac{3M^2\theta}{h^2}$$

and we get

$$\begin{aligned}\frac{M}{h^2}\left\{1+e\cos\left(\theta-\frac{3M^2}{h^2}\theta\right)\right\} &= \frac{M}{h^2}\left\{1+e\left(\cos\theta\cos\left(\frac{3M^2}{h^2}\theta\right)\right.\right.\\ &\qquad\left.\left.+\sin\theta\sin\left(\frac{3M^2}{h^2}\theta\right)\right)\right\}\\ &\approx \frac{M}{h^2}\left\{1+e\cos\theta+e\frac{3M^2}{h^2}\theta\sin\theta\right\}\approx u.\end{aligned}$$

So u has a maximum (and $r = 1/u$ has a minimum; i.e., we are at perihelion) when $\cos\left(\theta - \frac{3M^2}{h^2}\theta\right)$ is at a maximum. This occurs for $\theta = 0$ and

$$\theta = \frac{2\pi}{1 - 3M^2/h^2} \approx 2\pi\left(1 + \frac{3M^2}{h^2}\right)$$

(since $(1-x)^{-1} \approx 1 + x$ for $x \approx 0$). So the perihelion advances (in the direction of the orbital motion) by an amount $6\pi M^2/h^2$ per revolution.

Note. If we want the orbital precession per (Earth) century, we have

$$\Delta\theta_{\text{cent}} = n\Delta\theta = \frac{6\pi M^2 n}{h^2} = \frac{6\pi M n}{a(1-e^2)} \text{ (radians)}$$

(since $h^2/M = a(1-e^2)$ see page 195) where n is the number of orbits of the Sun that a planet makes per century.

Note. Table III-2 (reproduced below) gives the calculated precessions as observed and as predicted for 4 solar system objects. The last two columns represent precessions measured in seconds per century:

Planet	$a(\div 10^{11}\text{cm})$	e	n	General Relativity	Observed
Mercury	57.91	0.2056	415	$43.03''$	$43.11'' \pm 0.45''$
Venus	108.21	0.0068	149	$8.6''$	$8.4'' \pm 4.8''$
Earth	149.60	0.0167	100	$3.8''$	$5.0'' \pm 1.2''$
Icarus	161.0	0.827	89	$10.3''$	$9.8'' \pm 0.8''$

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