Proposition 0.8

Proposition 0.8. Any neighborhood \( A \) of \( x \in \overline{B} \) has a nonempty intersection with \( B \).

Proof. Let \( \Omega \subset A \) be an open neighborhood of \( x \) (which exists by the definition of "neighborhood"). Assume \( \Omega \cap B = \emptyset \). Then \( E \setminus \Omega \) is closed and \( B \subset E \setminus \Omega \). So (by the definition of closure) \( \overline{B} \subset E \setminus \Omega \). But \( x \in \Omega \) so \( x \in \overline{B} \), contradicting the hypothesis that \( x \in \overline{B} \). So the assumption that \( \Omega \cap B = \emptyset \) is false and hence \( \Omega \) has a nonempty intersection with \( B \) and, since \( \Omega \subset A \), \( A \) has a nonempty intersection with \( B \), as claimed. \( \square \)

Theorem 0.11

Theorem 0.11. The image by a continuous map of a compact set is compact.

Proof. Let \( K \subset E \) be a compact set. Let \( \{\Omega_i\}_{i \in I} \) be an open covering of \( f(K) \). Since \( f \) is continuous then, by the definition of "continuous," each \( f^{-1}(\Omega_i) \) is open in \( E \) and so \( \{f^{-1}(\Omega_i)\}_{i \in I} \) is an open covering of \( K \). Since \( K \) is compact then, by definition of "compact," there is finite set \( J \subset I \) such that \( K \subset \bigcup_{i \in J} f^{-1}(\Omega_i) \). So \( \{\Omega_i\}_{i \in J} \) is a finite subcovering of \( f(K) \). Since \( \{\Omega_i\}_{i \in I} \) is an arbitrary open covering of \( f(K) \), then \( f(K) \) is compact as claimed. \( \square \)

Theorem 0.1.1

Theorem 0.1.1. Let \( E \) and \( F \) be a Hausdorff topological spaces. If \( E \) is compact then any continuous \( f : E \rightarrow F \) is proper.

Proof. Let \( K \subset F \) be compact. Then by Theorem 0.9, \( K \) is closed. By Note 0.1.A, \( f^{-1}(K) \) is a closed subset of \( E \). Since \( E \) is compact, by Theorem 0.9 \( f^{-1}(K) \) is compact. Therefore, by the definition of "proper," \( f \) is proper as claimed. \( \square \)