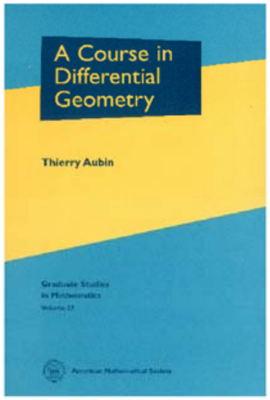
Differential Geometry

Chapter 0. Background Material

0.1. Topology—Proofs of Theorems



Proposition 0.8

Proposition 0.8. Any neighborhood A of $x \in \overline{B}$ has a nonempty intersection with B.

Proof. Let $\Omega \subset A$ be an open neighborhood of x (which exists by the definition of "neighborhood"). ASSUME $\Omega \cap B = \varnothing$. Then $E \setminus \Omega$ is closed and $B \subset E \setminus \Omega$. So (by the definition of closure) $\overline{B} \subset E \setminus \Omega$. But $x \in \Omega$ so $x \in \overline{B}$, CONTRADICTING the hypothesis that $x \in \overline{B}$. So the assumption that $\Omega \cap B = \varnothing$ is false and hence Ω has a nonempty intersection with B and, since $\Omega \subset A$, A has a nonempty intersection with B, as claimed. □

Theorem 0.11

Theorem 0.11. The image by a continuous map of a compact set is compact.

Proof. Let $K \subset E$ be a compact set. Let $\{\Omega_i\}_{i \in I}$ be an open covering of $f(K)$. Since f is continuous then, by the definition of "continuous," each $f^{-1}(\Omega_i)$ is open in E and so $\{f^{-1}(\Omega_i)\}_{i \in I}$ is an open covering of K. Since K is compact then, by definition of "compact," there is finite set $J \subset I$ such that $K \subset \cup_{i \in J} f^{-1}(\Omega_i)$. So $\{\Omega_i\}_{i \in J}$ is a finite subcovering of $f(K)$. Since $\{\Omega_i\}_{i \in I}$ is an arbitrary open covering of $f(K)$, then $f(K)$ is compact as claimed. □

Theorem 0.1.A

Theorem 0.1.A. Let E and F be a Hausdorff topological spaces. If E is compact then any continuous $f : E \to F$ is proper.

Proof. Let $K \subset F$ be compact. Then by Theorem 0.9, K is closed. By Note 0.1.A, $f^{-1}(K)$ is a closed subset of E. Since E is compact, by Theorem 0.9 $f^{-1}(K)$ is compact. Therefore, by the definition of "proper," f is proper as claimed. □