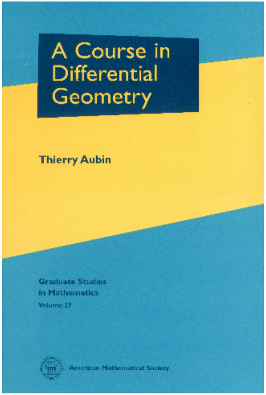
Differential Geometry

Chapter 0. Background Material

0.1. Topology—Proofs of Theorems



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Proposition 0.8

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Theorem 0.11

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Proof. Let $K \subset E$ be a compact set. Let $\{\Omega_i\}_{i \in I}$ be an open covering of $f(K)$. Since f is continuous then, by the definition of "continuous," each $f^{-1}(\Omega_i)$ is open in E and so $\{f^{-1}(\Omega_i)\}_{i \in I}$ is an open covering of K.

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Theorem 0.1.A

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Proof. Let $K \subset F$ be compact. Then by Theorem 0.9, K is closed. By Note 0.1.A, $f^{-1}(K)$ is a closed subset of E. Since E is compact, by Theorem 0.9 $f^{-1}(K)$ is compact. Therefore, by the definition of "proper," f is proper as claimed. □

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