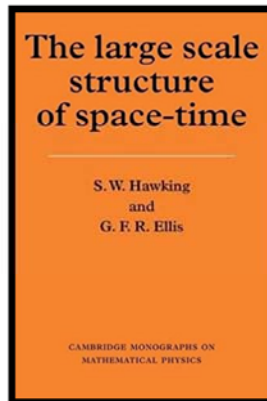


Differential Geometry

Chapter 2. Differential Geometry

2.1. Manifolds—Proofs of Theorems



Theorem 2.1.A

Theorem 2.1.A. For \mathcal{M} a manifold and $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ a complete atlas, each $\varphi_\alpha : \mathcal{U}_\alpha \rightarrow \varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$ is a homeomorphism.

Proof. Recall that for X and Y topological spaces, $f : X \rightarrow Y$ a bijection is a homeomorphism if both f and $f^{-1} : Y \rightarrow X$ are continuous. Let \mathcal{O} be a set open in \mathcal{U}_α under the subspace topology on \mathcal{M} (see Section 16, “The Subspace Topology” of my online topology notes at <http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-16.pdf>). Then $\mathcal{O} = \mathcal{U}_\alpha \cap U$ where U is an open set in the topology on \mathcal{M} . Since $\{\mathcal{U}_\beta\}$ is a basis for the topology on \mathcal{M} then $U = \cup_{\beta \in B} \mathcal{U}_\beta$ for some set of index values B . Then

$$\mathcal{O} = \mathcal{U}_\alpha \cap (\cup_{\beta \in B} \mathcal{U}_\beta) = \cup_{\beta \in B} (\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

For one to one function f we have $f(A \cap C) = f(A) \cap f(C)$ for any sets A and C (see J. R. Kirkwood's *An Introduction to Analysis*, 2nd Edition, PWS Publishing Company and Waveland Press, Inc. (1995), Exercise 1.1.14).

Theorem 2.1.A (continued 1)

Proof (continued). Also, for any function f defined on a set C we have $f(\cup_{\gamma \in G} C_\gamma) = \sup_{\gamma \in G} f(C_\gamma)$ where $C_\gamma \subset C$ for all $\gamma \in G$ (see Kirkwood's Exercise 1.1.13(a) and the note that follows it on page 13). So

$$\varphi_\alpha(\mathcal{O}) = \varphi_\alpha(\mathcal{U}_\alpha \cap U) = \varphi_\alpha(\cup_{\beta \in B} (\mathcal{U}_\alpha \cap \mathcal{U}_\beta)) = \cup_{\beta \in B} \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

Notice that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is open in the subspace topology and φ_α restricted to $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $\varphi_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$, gives a chart $(\varphi_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}, \mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ which is compatible with the other charts on \mathcal{M} and so is included in the complete atlas. It then follows that $\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is open in \mathbb{R}^n . Therefore $\cup_{\beta \in B} \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = \varphi_\alpha(\mathcal{O})$ is open in \mathbb{R}^n . Now \mathcal{O} is an arbitrary open subset of \mathcal{U}_α and we now have that the inverse image (with respect to the function φ_α^{-1}), $(\varphi_\alpha^{-1})^{-1}(\mathcal{O}) = \mathcal{O}$ is open (since φ_α is one to one, nothing is mapped by φ_α to $\varphi_\alpha(\mathcal{O})$ other than elements of \mathcal{O}). Hence function φ_α^{-1} is continuous.