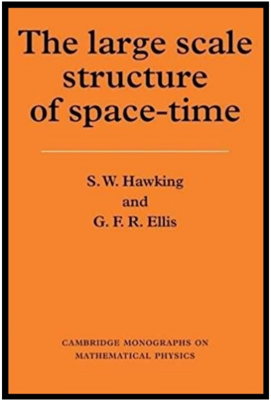
Differential Geometry

Chapter 2. Differential Geometry

2.1. Manifolds—Proofs of Theorems



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Theorem 2.1.A

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Proof. Recall that for X and Y topological spaces, $f : X \to Y$ a bijection is a homeomorphism if both f and $f^{-1} : Y \to X$ are continuous. Let $\mathcal{O}$ be a set open in $\mathcal{U}_\alpha$ under the subspace topology on $\mathcal{M}$ (see Section 16, "The Subspace Topology" of my online topology notes at http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-16.pdf).

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$$\mathcal{O} = \mathcal{U}_\alpha \cap (\cup_{\beta \in B} \mathcal{U}_\beta) = \cup_{\beta \in B} (\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

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For one to one function f we have $f(A \cap C) = f(A) \cap f(C)$ for any sets A and C (see J. R. Kirkwood's *An Introduction to Analysis*, 2nd Edition, PWS Publishing Company and Waveland Press, Inc. (1995), Exercise 1.1.14).

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Theorem 2.1.A (continued 1)

Proof (continued). Also, for any function f defined on a set C we have $f\left(\cup_{\gamma\in G} C_\gamma\right) = \sup_{\gamma\in G} f(C_\gamma)$ where $C_\gamma \subset C$ for all $\gamma \in G$ (see Kirkwood's Exercise 1.1.13(a) and the note that follows it on page 13). So

$$\varphi_\alpha(\mathcal{O}) = \varphi_\alpha(\mathcal{U}_\alpha \cap U) = \varphi_\alpha\left(\cup_{\beta\in B}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\right) = \cup_{\beta\in B}\varphi_\alpha\left(\mathcal{U}_\alpha \cap \mathcal{U}_\beta\right).$$

Notice that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is open in the subspace topology and φ_α restricted to $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $\varphi_\alpha|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta}$, gives a chart $\left(\varphi_\alpha|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta}, \mathcal{U}_\alpha \cap \mathcal{U}_\beta\right)$ which is compatible with the other charts on $\mathcal{M}$ and so is included in the complete atlas.

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Theorem 2.1.A (continued 2)

Proof (continued).

Theorem 2.1.A (continued 3)

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