Differential Geometry

Chapter 2. Manifolds and Tensor Fields

2.2. Vectors—Proofs of Theorems

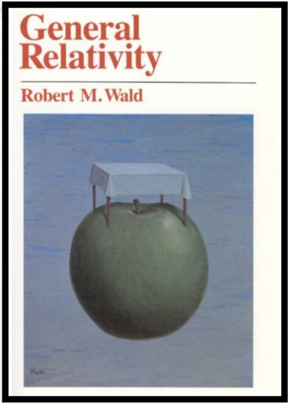

Table of contents

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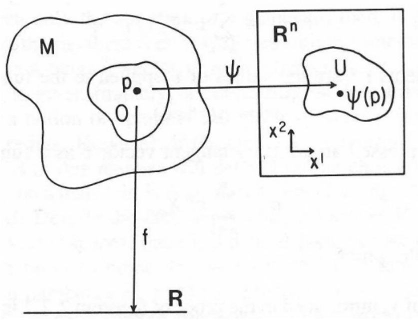

Figure 2.3 from Wald, page 15

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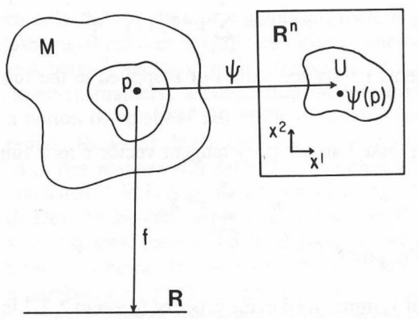

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Theorem 2.2.1 (continued 1)

Proof (continued). If $f \in \mathcal{F}$ then by the definition of "C^∞ function" we have $f \circ \psi^{-1} : U \to \mathbb{R}$ is C^∞. (We defined $f : M \to M'$ as C^∞ in Section 2.1 and involved $\psi'_\beta : M' \to \mathbb{R}^n$, but here $M' = \mathbb{R}$ so we take ψ'_β as the identity and $\psi'_\beta \circ f \circ \psi^{-1} = f \circ \psi^{-1}$.) For $\mu = 1, 2, \ldots, n$ define $X_\mu : \mathcal{F} \to \mathbb{R}$ by

$$X_\mu(f) = \frac{\partial}{\partial x^\mu} \left[f \circ \psi^{-1} \right] \Big|_{\psi(p)}$$

where $(x^1, x^2, \ldots, x^\mu)$ are the Cartesian coordinates of $\mathbb{R}^n$. Notice that $f \circ \psi^{-1} : U \to \mathbb{R}$ and $U \subset \mathbb{R}^n$, so in fact $f \circ \psi^{-1}$ is a function of $x^1, x^2, \ldots, x^n$.

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Theorem 2.2.1 (continued 2)

Proof (continued). Then

$$X_i(f_i) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

and so $(a_1 X_1 + a_2 X_2 + \cdots + a_n X_n)(f_i) = 0$ if and only if $a_i = 0$. So by applying $a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$ to $f_1, f_2, \ldots, f_n$ and setting each equal to 0 implies that $a_1 = a_2 = \cdots = a_n = 0$. So $X_1, X_2, \ldots, X_n$ are linearly independent.

By Problem 2.2, if $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞, then for each $a = (a^1, a^2, \ldots, a^n) \in \mathbb{R}^n$, there exists C^∞ functions H_μ such that for all $x \in \mathbb{R}^n$ we have

$$F(x) = F(a) + \sum_{\mu=1}^{n} (x^\mu - a^\mu) H_\mu(x) \text{ and } H_\mu(a) = \left. \frac{\partial F}{\partial x^\mu} \right|_{x=a}. \quad (2.2.3/2.2.4)$$

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By Problem 2.2, if $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞, then for each $a = (a^1, a^2, \ldots, a^n) \in \mathbb{R}^n$, there exists C^∞ functions H_μ such that for all $x \in \mathbb{R}^n$ we have

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Theorem 2.2.1 (continued 3)

Proof (continued). We take $F = f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}$ and $a = \psi(p)$ to get from Problem 2.2 that for all $q \in O$ (where $\psi(q) = x \in \mathbb{R}^n$; think of both x and q as variables)

$$F(x) = (f \circ \psi^{-1})(\psi(q)) = f(q)$$

$$= F(\psi(p)) + \sum_{\mu=1}^{n} [x^\mu \circ \psi(q) - x^\mu \circ \psi(p)] H_\mu(\psi(q))$$

where $x^\mu \circ \psi(q)$ denotes the μ-th coordinate of $\psi(q) \in \mathbb{R}^n$. Also, $F(\psi(p)) = (f \circ \psi^{-1})(\psi(p)) = f(p)$, so

$$f(q) = f(p) + \sum_{\mu=1}^{n} (x^\mu \circ \psi(q) - x^\mu \circ \psi(p)) H_\mu(\psi(q)). \qquad (2.2.4)$$

Let $v \in V_p$. We now show that v is a linear combination of $X_1, X_2, \ldots, X_n$ (and hence $X_1, X_2, \ldots, X_n$ is a basis for V_p).

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Theorem 2.2.1 (continued 4)

Proof (continued). Let $f \in \mathcal{F}$. We have

$$\begin{aligned} v(f) &= v(f(q))\big|_{q=p} \\ &= v\left[f(p) + \sum_{\mu=1}^{n}[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)]H_\mu(\psi(q))\right] \text{ by (2.2.4)} \\ &= v[f(p)] + \sum_{\mu=1}^{n} v\left[[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)]H_\mu(\psi(q))\right] \\ &\quad \text{since } v \text{ is linear} \\ &= v[f(p)] + \sum_{\mu=1}^{n}\Big\{ (x^\mu \circ \psi(q) - x^\mu \circ \psi(p))\big|_{q=p} v[H_\mu(\psi(q))] \\ &\quad + v[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)]H_\mu(\psi(q))\big|_{q=p}\Big\} \text{ by Leibniz's Rule} \end{aligned}$$

Theorem 2.2.1 (continued 5)

Proof (continued).

$$\begin{aligned} V(f) &= 0 + \sum_{\mu=1}^{n} \{0 + v[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)] H_\mu(\psi(p))\} \\ &\quad \text{since } f(p) \text{ is constant} \\ &= \sum_{\mu=1}^{n} v[(x^\mu \circ \psi)(q)](H_\mu \circ \psi)(p) \text{ since } (x^\mu \circ \psi)(p) \text{ is constant.} \end{aligned}$$

But by equation (2.2.3),

$$H_\mu \circ \psi(p) = H_\mu(a) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x=a} = \left. X_\mu(f)\right|_{x=a}.$$

So

$$v(f) = \sum_{\mu=1}^{n} v((x^\mu \circ \psi)(q)) \left. X_\mu(f)\right|_{x=a}.$$

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$$V(f) = 0 + \sum_{\mu=1}^{n} \{0 + v[x^\mu \circ \psi(q) - x^\mu \circ \psi(p)] H_\mu(\psi(p))\}$$

since $f(p)$ is constant

$$= \sum_{\mu=1}^{n} v[(x^\mu \circ \psi)(q)](H_\mu \circ \psi)(p) \text{ since } (x^\mu \circ \psi)(p) \text{ is constant.}$$

But by equation (2.2.3),

$$H_\mu \circ \psi(p) = H_\mu(a) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x=a} = X_\mu(f)|_{x=a}.$$

So

$$v(f) = \sum_{\mu=1}^{n} v((x^\mu \circ \psi)(q)) \, X_\mu(f)|_{x=a}.$$

Theorem 2.2.1 (continued 6)

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Proof (continued). With v^μ set equal to $v((x^\mu \circ \psi)(q))$ we have

$$v(f) = \left(\sum_{\mu=1}^{n} v^\mu X_\mu \right)(f)$$

and so $v \sum_{\mu=1}^{n} v^\mu X_\mu$ and $X_1, X_2, \ldots, X_n$ is a spanning set for V_p. Therefore, $X_1, X_2, \ldots, X_n$ is a basis for V_p. □