Chapter 2. Differential Geometry

2.1. Manifolds (Partial)

Note. Hawking and Ellis describe a manifold as:

> "A manifold is essentially a space which is locally similar to Euclidean space in that it can be covered by coordinate patches." [page 11]

We will discuss different "coordinate systems" on manifolds (such as Cartesian coordinates and polar coordinates, both of which are defined on $\mathbb{R}^2$), but the properties of manifolds which we study will be independent of the choice of a coordinate system.

Note. We let $\mathbb{R}^n$ denote the Euclidean space of n-dimensions and give it the usual topology; that is, the topology induced by the Euclidean metric d where

$$d((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)) = ((x^1 - y^1)^2 + (x^2 - y^2)^2 + \cdots + (x^n - y^n)^2)^{1/2}.$$

Notice that we use superscripts to indicate coordinates of points instead of subscripts.

Definition. Let $\mathcal{O} \subset \mathbb{R}^n$ and $\mathcal{O}' \subset \mathbb{R}^m$ be open sets. A map $\varphi : \mathcal{O} \to \mathcal{O}'$ is *in the class* C^r if for each point $p = (x^1, x^2, \ldots, x^n) \in \mathcal{O}$, the coordinates of $\varphi(p) \in \mathcal{O}'$, say $\varphi(p) = (x^{1\prime}, x^{2\prime}, \ldots, c^{m\prime})$, are r-times continuously differentiable functions of $x^1, x^2, \ldots, x^n$. If such a function is C^r for all $r \geq 0$ then the function is *in the class* C^∞. A continuous function mapping $\mathcal{O}$ to $\mathcal{O}'$ is *in the class* C^0.

Example. Let $\mathcal{O} = \mathbb{R}^2$ and $\mathcal{O}' = \mathbb{R}^3$ and define $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ as

$$\varphi(p) = \varphi((x^1, x^2)) = ((x^1)^2 x^2, \sin x^2, x^1 e^{(x^2)}).$$

Then the coordinates of $\varphi(p)$ are $x^{1\prime} = (x^1)^2 x^2$, $x^{2\prime} = \sin x^2$, and $x^{3\prime} = x^1 e^{(x^2)}$. Since $x^{1\prime}$, $x^{2\prime}$, and $x^{3\prime}$ are differentiable functions of x^1 and x^2 (of all orders), then $\varphi \in C^\infty$.

Note. Hawking and Ellis do not distinguish between points in $\mathbb{R}^n$ and vectors in $\mathbb{R}^n$ so they appear to be adding points together in places (when in fact, they are dealing with vector sums). So for "point" $p \in (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ they have $|p| = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}$. We'll follow this notation in these notes, though we would normally think of p as a *vector* in $\mathbb{R}^n$ and we would call the quantity $|p|$ the *norm* of p (normally denoted $\|p\|$).

Note. In what follows, the word "function" is used to mean a mapping of some subset of $\mathbb{R}^n$ into $\mathbb{R}$. A "map" is a mapping of some subset of $\mathbb{R}^n$ into $\mathbb{R}^m$.

Definition. Let $\mathcal{O} \subset \mathbb{R}^n$ be open. Function $f : \mathcal{O} \to \mathbb{R}$ is *locally Lipschitz* on $\mathcal{O}$ if for each open set $\mathcal{U} \subset \mathcal{O}$ with compact closure in $\mathcal{O}$ there is constant $K \in \mathbb{R}$ such that for any $p, q \in \mathcal{U}$ we have $|f(p) - f(q)| \le K|p - q|$.

Note. A locally Lipschitz function on $\mathcal{O} \subset \mathbb{R}^n$ is continuous on $\mathcal{O}$. This follows from the epsilon/delta definition of continuity (just let $\delta = \varepsilon/K$).

Definition. Let $\mathcal{O} \subset \mathbb{R}^n$ be open. A map $\varphi : \mathcal{O} \to \mathbb{R}^m$ is *locally Lipschitz*, denoted by C^{1-}, if for each point $p = (x^1, x^2, \ldots, x^n) \in \mathcal{O}$, the coordinates of $\varphi(p) \in \mathbb{R}^m$, say $\varphi(p) = (x^{1\prime}, x^{2\prime}, \ldots, x^{m\prime})$ are locally Lipschitz functions of $x^1, x^2, \ldots, x^n$. Similarly, define a map as C^{r-} if it is C^{r-1} and if the $(r-1)$th derivatives of the coordinates of $\varphi(p)$ are locally Lipschitz functions of the coordinates of p.

Note. For more details on Lipschitz functions mapping subsets of $\mathbb{R}$ into $\mathbb{R}$, see my online notes for Complex Analysis 1 (MATH 5510), "A Primer on Lipschitz Functions" at: `http://faculty.etsu.edu/gardnerr/5510/CSPACE.pdf`.

Note. We now formally define a "manifold" and give some examples.

Definition. A C^r *n-dimensional manifold* $\mathcal{M}$ is a set $\mathcal{M}$ together with a C^r *atlas* $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ of *charts* $(\mathcal{U}_\alpha, \varphi_\alpha)$ where $\mathcal{U}_\alpha$ are subset of $\mathcal{M}$ and φ_α are one to one maps of the corresponding $\mathcal{U}_\alpha$ to open sets in $\mathbb{R}^n$ such that:

(1) the $\mathcal{U}_\alpha$ cover $\mathcal{M}$; that is, $\mathcal{M} = \cup_\alpha \mathcal{U}_\alpha$,

(2) if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is nonempty then the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is a C^r map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

Note. The mappings of part (2) of the definition of manifold are illustrated in Hawking and Ellis' Figure 4 of page 12:

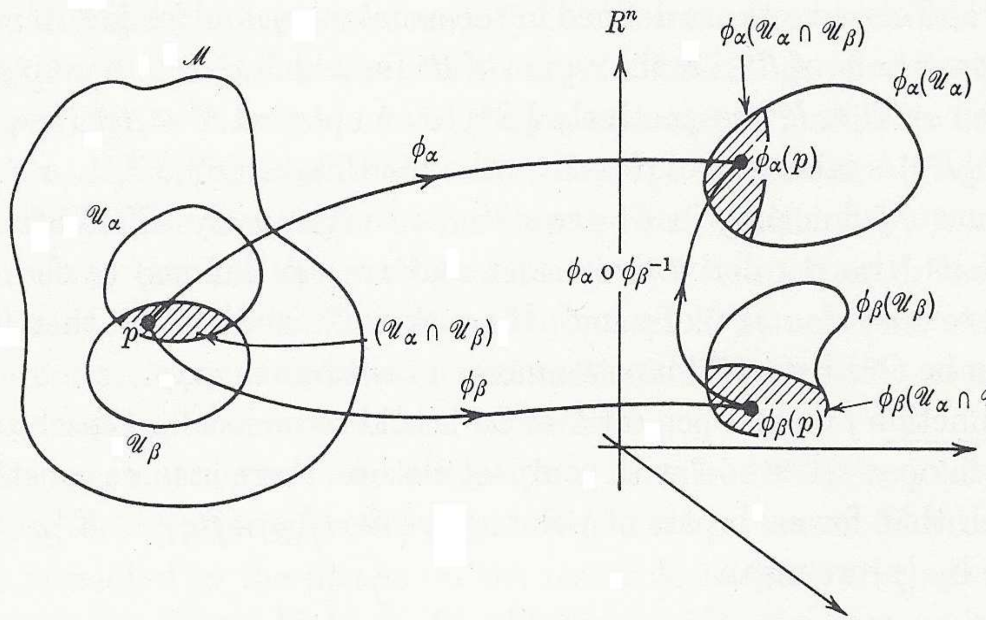


FIGURE 4. In the overlap of coordinate neighbourhoods $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$, coordinates are related by a C^r map $\phi_\alpha \circ \phi_\beta^{-1}$.

Definition. A C^r atlas on manifold $\mathcal{M}$ is *compatible* with a given C^r atlas on $\mathcal{M}$ if their union is a C^r atlas on $\mathcal{M}$. The atlas consisting of all atlases compatible with a given atlas is the *complete atlas* for the given atlas.

Note. Hawking and Ellis say that a complete atlas is the set of all possible coordinate systems covering manifold $\mathcal{M}$. Wald in his *General Relativity* addresses this by requiring the collection of charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ to be maximal; by convention, Wald includes this in his definition of manifold (see has page 12).

Note. A basis for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that:

(1) for each $x \in X$ there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$, and

(2) if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

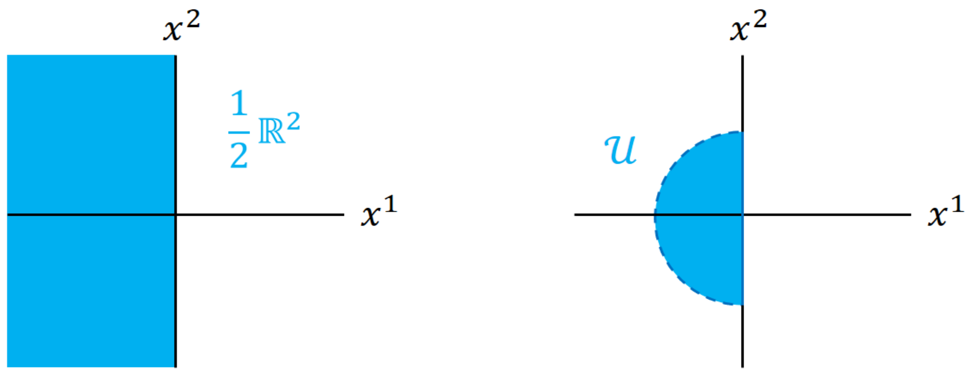
The topology $\mathcal{T}$ generated by $\mathcal{B}$ is defined as: A subset $U \subset X$ is in $\mathcal{T}$ if for each $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. See my notes from Munkres' *Topology*, 2nd Edition, for "Basis for a Topology" available online: `http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-13.pdf`. Lemma 13.1 of these online notes states: "Let X be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$." Notice that for a complete atlas $\{\mathcal{U}_\alpha, \varphi_\alpha\}$, the sets $\mathcal{U}_\alpha$ satisfy the definition of a basis for a topology on $\mathcal{M}$. So we put a topology on $\mathcal{M}$ generated by $\{\mathcal{U}_\alpha\}$. So the open sets in the topology are, by Lemma 13.1 mentioned above, unions of elements of $\{\mathcal{U}_\alpha\}$. Notice that each $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n$ is, by definition, one to one and maps $\mathcal{U}_\alpha$ to an open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$.

Theorem 2.1.A. For $\mathcal{M}$ a manifold and $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ a complete atlas, each $\varphi_\alpha : \mathcal{U}_\alpha \to \varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$ is a homeomorphism.

Definition. With $\mathbb{R} = \{(x^1, x^2, \ldots, x^n) \mid -\infty < x^i < \infty \text{for } i = 1, 2, \ldots, n\}$, we define the *lower half of* $\mathbb{R}^n$ as

$$\frac{1}{2}\mathbb{R}^n = \{(x^1, x^2, \ldots, x^n) \mid x^1 \leq 0, -\infty < x^i < \infty \text{ for } i = 2, 3, \ldots, n\}.$$

Note. With $n = 2$, $\frac{1}{2}\mathbb{R}^2$ is the closed left half-plane. This is a subset of $\mathbb{R}^2$ and gets the subspace topology so that the open sets in $\frac{1}{2}\mathbb{R}^2$ are of the form $\mathcal{O} \cap \frac{1}{2}\mathbb{R}^2$ where $\mathcal{O}$ is open in $\mathbb{R}^2$. So we may have open sets in $\frac{1}{2}\mathbb{R}^2$ for example, of the form $\mathcal{U} = \{(x^2, x^2) \in \frac{1}{2}\mathbb{R}^2 \mid |(x^1, x^2)| < 1\}$:



So $\frac{1}{2}\mathbb{R}^2$ has the boundary $\{(x^1, x^2) \mid x^1 = 0, -\infty < x^2 < \infty\}$, and in general $\frac{1}{2}\mathbb{R}^n$ has the boundary

$$\{(x^1, x^2, \ldots, x^n) \mid x^1 = 0, -\infty < x^i < \infty \text{ for } i = 2, 3, \ldots, n\}.$$

We now define a manifold with a boundary using $\frac{1}{2}\mathbb{R}^n$, similar to our definition of manifold above.

Definition. A C^r *n-dimensional manifold* $\mathcal{M}$ *with a boundary* is a set $\mathcal{M}$ together with a C^r *atlas* $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ of *charts* $(\mathcal{U}_\alpha, \varphi_\alpha)$ where $\mathcal{U}_\alpha$ are subset of $\mathcal{M}$ and φ_α are one to one maps of the corresponding $\mathcal{U}_\alpha$ to open sets in $\frac{1}{2}\mathbb{R}^n$ such that:

(1) the $\mathcal{U}_\alpha$ cover $\mathcal{M}$; that is, $\mathcal{M} = \cup_\alpha \mathcal{U}_\alpha$,

(2) if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is nonempty then the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is a C^r map of an open subset of $\frac{1}{2}\mathbb{R}^n$ to an open subset of $\frac{1}{2}\mathbb{R}^n$.

The *boundary* of $\mathcal{M}$, denoted $\partial(\mathcal{M})$, is the set of all points of $\mathcal{M}$ whose image under a map φ_α lies on the boundary of $\frac{1}{2}\mathbb{R}^n$ in $\mathbb{R}^n$; that is,

$$\partial(\mathcal{M}) = \{m \in \mathcal{M} \mid \varphi(\alpha(m) = (0, x^2, x^3, \ldots, x^n) \text{ for some } \varphi_\alpha$$

$$\text{and for some } (0, x^2, x^3, \ldots, n^n) \in \mathbb{R}^n\}.$$

Lemma 2.1.A. $\frac{1}{2}\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^{n-1}$.

Theorem 2.1.B. For $\mathcal{M}$ an n-dimensional C^r manifold with a boundary, $\partial(\mathcal{M})$ is an $(n-1)$=dimensional C^r manifold without a boundary.

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