Chapter 0. Background Material

Note. In this chapter we briefly cover material on topology, tensors, and differential calculus. When I have online notes for a topic or result, we will refer to those notes (which will be the case for most often for topology and analysis results).

0.1. Topology

Note. In this section we define a topology and give a few additional definitions. I have extensive online notes (at the undergraduate/graduate "cross-listed" level) on topology at: "Introduction to Topology Class Notes: General Topology".

Definition 0.1. A *topology* on a point set E is a family $\mathcal{O}$ of subsets of E, called *open sets*, such that:

(a) The set E and the null set $\varnothing$ are open sets.

(b) Any union of open sets is an open set.

(c) Any finite intersection of open sets is an open set.

The pair $(E, \mathcal{O})$ is a *topological space*.

Example/Definition. Let $E = \mathbb{R}^n$. With d as the Euclidean metric on $\mathbb{R}^n$, define the *open ball*

$$B_x(\rho) = \{y \in \mathbb{R}^n \mid d(x, y) < \rho\}$$

with center x and radius ρ. If we take $\mathcal{O}$ as the collection of all $X \subset \mathbb{R}^n$ such that for all $x \in x$ there is $\rho > 0$ such that $B_x(\rho) \subset X$, then $(\mathbb{R}^n, \mathcal{O})$ is a topological space. The topology $\mathcal{O}$ is the *usual topology* on $\mathbb{R}^n$.

Definition 0.3. Let $(E,)$ be a topological space and let $F \subset E$. The *induced topology* on F is the set $\tilde{\mathcal{O}} = \{A \cap F \mid A \in \mathcal{O}\}$.

Definition 0.5. A *neighborhood* of a point x in a topological space E is a subset of E containing an open set which contains point x.

Note. Often times the term "neighborhood" is used to mean an open neighborhood, but that is not the case here. Instead here a neighborhood of x contains an open neighborhood of x.

Definition. Let $(E, \mathcal{O})$ be a topological space Set $B \subset E$ is *closed* if $E \setminus B$ is open The topological space is *connected* if the only subsets which are both open and closed are $\varnothing$ and E. The *closure* of $B \subset E$ is the intersection of all closed sets containing B, denoted $\overline{B}$.

Note/Definition. We might define a *separation* in a topological space as two nonempty sets $X, Y \subset E$ where $X \cap Y = \varnothing$, $X \cup Y = E$, and both X and Y are open (of course $X = E \setminus Y$ so both sets are also closed). For this concept in the setting of $\mathbb{R}$, see my online notes on "Topology of the Real Numbers" (see page 11).

Proposition 0.8. Any neighborhood A of $x \in \overline{B}$ has a nonempty intersection with B.

Definition 0.7. The *interior* of $B \subset E$, denoted $\mathring{B}$, is the union of all open subsets of B.

Note/Definition. Recall that a *basis* for a topology $\mathcal{O}$ is a collection $\mathcal{B}$ of subsets of E such that:

(1) For each $x \in E$ there is at least one element $B \in \mathcal{B}$ with that $x \in B$.

(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_1 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

A topological space is *separable* if it has a countable basis.

Definition. A topological space is *Hausdorff* if any two distinct points have disjoint neighborhoods.

Definition. A family of subset of E, $\{\Omega_i\}_{\in I}$, is a *covering* of a set $B \subset E$ if $B \subset \cup_{i\in I}\Omega_i$. A *subcovering* of this covering is a subset $\{\Omega_i\}_{i\in J}$ where $J \subset I$ which is itself a covering. If J is finite then the subcovering is *finite*.

Definition 0.8. A subset $A \subset E$ is a *compact set* if it is Hausdorff (that is, if the induced topology on A is Hausdorff) and if any covering of A by open sets has a finite subcovering.

Note. In Munkres' Section 26, "Compact Sets", the definition of "compact" does not include the condition of Hausdorff.

Note. The following theorem is a combination of Munkres' Theorem 26.2 and Theorem 26.3.

Theorem 0.9. Let E be a Hausdorff topological space. If $K \subset E$ is a compact set, then K is closed. If E is also compact, then $K \subset E$ is compact if K is closed.

Definition 0.10. Let E and F be two topological spaces. A map $f : E \to F$ is *continuous* if the preimage $f^{-1}(\Omega)$ of any open set $\Omega \subset F$ is an open subset of E.

Note 0.1.A. We can also show that if $f : E \to F$ is continuous then the preimage $f^{-1}(\Omega)$ of any closed set $\Omega \subset F$ is a closed subset of E.

Theorem 0.11. The image by a continuous map of a compact set is compact.

Definition 0.12. A continuous map is said to be *proper* if the preimage of every compact set is a compact set.

Theorem 0.1.A. Let E and F be a Hausdorff topological spaces. If E is compact then any continuous $f : E \to F$ is proper.

Note. As with isomorphisms of various algebraic structures, we are interested in when two topological spaces are structurally the same. The relevant mapping is a homeomorphism.

Definition. Let E and F be two topological spaces. A one to one and onto map $f : E \to F$ is a *homeomorphism* if both f and f^{-1} are continuous.

Note. Since a homeomorphism is one to one and onto, then E and F must be the same cardinality. Since f and f^{-1} are continuous then the open sets in one topological space correspond to the open sets in the other topological space.