0.2. Tensors

Note. In this section we define the tensor product of two finite dimensional vector spaces, tensor, covariant, contrapositive, and illustrate these ideas with examples.

Definition 0.13. Let $E$ and $F$ be two real vector spaces of dimensions $n$ and $p$, respectively. The tensor product of $E$ and $F$ is a vector space of dimension $np$, denoted $E \otimes F$. A vector of $E \otimes F$ is called a tensor. For $x \in E$ and $y \in F$ we associate $x \otimes y \in E \otimes F$. This product has the following properties:

(a) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$ and $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$ where $x, x_1, x_2 \in E$ and $y, y_1, y_2 \in F$.

(b) If $\alpha \in \mathbb{R}$ is a scalar then

$$(\alpha x) \otimes y = x \otimes (\alpha y) = \alpha(x \otimes y).$$

(c) If $\{e_i\}_{1 \leq i \leq n}$ is a basis of $E$ and $\{f_j\}_{1 \leq j \leq p}$ is a basis of $F$, then $e_i \otimes f_j$ is a basis of $E \otimes F$.

(d) If $G$ is a third real vector space with $E \otimes (F \otimes G) = (E \otimes F) \otimes G$, then

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

for all $z \in G$.

Note. By part (d) of the definition, the product of tensors is an associative operation. However, we have not assumed commutativity.
Note/Definition. We follow the Einstein convention when dealing with summations. When the index (such as $i$ or $j$) is present above and below (that is, when the index is present as both a superscript and a subscript) then summation over the index is assumed. In $E \otimes F$ where where $E$ is dimension $n$ and $F$ is dimension $p$, $i$ is summed from 1 to $n$ and $j$ is summed from 1 to $p$.

Example. Let $x \in E$ and $y \in F$ where

$$x = \sum_{i=1}^{n} x^i e_i = x^i e_i \text{ and } y = \sum_{j=1}^{p} y^j f_j = y^j f_j.$$ 

By (a) in the definition of product of tensors,

$$x \otimes y = (x^i e_i) \otimes (y^j f_j) = \left( \sum_{i=1}^{n} x^i e_i \right) \otimes \left( \sum_{j=1}^{p} y^j f_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{p} (x^i y^j)(e_i \otimes f_j) = x^i y^j e_i \otimes f_j.$$ 

So the components of $x \otimes y$ with respect to the basis $\{e_i \otimes f_j\}$ (for $1 \leq i \leq n$ and $1 \leq j \leq p$) of $E \otimes F$ are $t^{ij} = x^i y^j$.

Note. Suppose that both $\{e_i\}_{1 \leq i \leq n}$ and $\{\tilde{e}_\alpha\}_{1 \leq \alpha \leq n}$ are bases for $E$ and that both $\{f_j\}_{1 \leq j \leq p}$ and $\{\tilde{f}_\beta\}_{1 \leq \beta \leq p}$ are bases for $F$. Then we can write

$$\tilde{e}_\alpha = \sum_{i=1}^{n} a^i_\alpha e_i = a^i_\alpha e_i \text{ and } \tilde{f}_\beta = \sum_{\beta=1}^{p} c^j_\beta f_j = c^j_\beta f_j,$$

or

$$e_i = \sum_{\alpha=1}^{n} b^i_\alpha \tilde{e}_\alpha = b^i_\alpha \tilde{e}_\alpha \text{ and } f_j = \sum_{\beta=1}^{p} d^j_\beta \tilde{f}_\beta = d^j_\beta \tilde{f}_\beta.$$
Here, the matrices \((a^i_\alpha)\) (with \(1 \leq i, \alpha \leq n\)) and \((b^\alpha_i)\) (with \(1 \leq i, \alpha \leq n\)) are inverses of each other, and matrices \((c^j_\beta)\) (with \(1 \leq j, \beta \leq p\)) and \((d^\beta_j)\) (with \(1 \leq j, \beta \leq p\)) are inverses of each other:

\[
(b^\alpha_i)(a^i_\alpha) = \begin{pmatrix}
  b^1_1 & b^2_1 & \cdots & b^n_1 \\
  b^1_2 & b^2_2 & \cdots & b^n_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  b^1_n & b^2_n & \cdots & b^n_n
\end{pmatrix}
\begin{pmatrix}
  a^1_1 & a^2_1 & \cdots & a^n_1 \\
  a^1_2 & a^2_2 & \cdots & a^n_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  a^1_n & a^2_n & \cdots & a^n_n
\end{pmatrix}
\]

where \(e_i = b^\alpha_i \tilde{e}_\alpha = b^\alpha_i (a^j_\alpha e_j) = b^\alpha_i a^j_\alpha e_j\) so that \(b^\alpha_i a^j_\alpha = \delta^j_i = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}\) and hence \((b^\alpha_i)(a^i_\alpha) = I\). Here, \(\delta^j_i\) is called the Kronecker tensor. Similarly, \((a^j_\alpha)(b^i_\alpha) = I\) (or this follows since the matrices are square; see “Theorem 1.11. A Commutative Property” in my online notes for 1.5. Inverses of Square Matrices). So for tensor

\[
T = t^{\alpha\beta} \tilde{e}_\alpha \otimes \tilde{f}_\beta = t^{ij} e_i \otimes f_j
\]

we have

\[
T = t^{ij} e_i \otimes f_j = t^{ij} (b^\alpha_i \tilde{e}_\alpha) \otimes (d^\beta_j \tilde{f}_\beta) = t^{ij} b^\alpha_i d^\beta_j \tilde{e}_\alpha \otimes \tilde{f}_\beta = t^{\alpha\beta} \tilde{a}_\alpha \otimes \tilde{f}_\beta
\]

so that we can relate the coordinates of \(T\) with respect to the different bases as

\[
t^{\alpha\beta} = t^{ij} b^\alpha_i d^\beta_j\]

and similarly \(t^{ij} = t^{\alpha\beta} a^i_\alpha c^j_\beta\).
**Note/Definition.** Recall that the collection of linear mappings between (finite dimensional real) vector spaces \( X \) and \( Y \) form a vector space themselves, denoted \( L(X,Y) \) (see page 24 of C.T.J Dodson and T. Poston’s *Tensor Geometry: The Geometric Viewpoint and its Uses*, 2nd Edition, Graduate Texts in Mathematics #130, Springer Verlag (1991)). In fact, if \( X \) is dimension \( m \) and \( Y \) is dimension \( n \), so that \( X \cong \mathbb{R}^n \) and \( Y \cong \mathbb{R}^m \) (by the Fundamental Theorem of Finite Dimensional Vector Spaces; see my online notes for 3.3 Coordinatization of Vectors, page 5), then each element of \( L(X,Y) \) is represented by a \( m \times n \) matrix (see my online notes for 3.4. Linear Transformations, “Theorem 3.10. Matrix Representations of Linear Transformations” on page 10). Here, we interpret the vectors in \( \mathbb{R}^n \) and \( \mathbb{R}^m \) as column vectors. In the event that \( Y = \mathbb{R} \), an element of \( L(X,\mathbb{R}) \) ix called a *linear functional* (or *dual vector* or *covariant vector*). The vector space \( L(X,\mathbb{R}) \) is the *dual space* of \( X \), denoted \( X^* \) (see page 57 of *Tensor Geometry*). With \( X = \mathbb{R}^n \), the vector space of \( n \)-dimensional column vectors, the dual space is \( X^* \) the vector space of \( n \)-dimensional row vectors. We then get the functional action produced by multiplying \( n \times 1 \) vector \( x \in X \) on the left by some \( 1 \times n \) vector \( y \in X^* \), producing effectively a dot product. We should note that above, \( m \times n \) matrix \( A \) is applied to \( n \)-dimensional column vector \( x \) to produce \( m \)-dimensional column vector \( Ax \in Y \) (otherwise we take row vectors in \( \mathbb{R}^n \) and \( \mathbb{R}^m \) and an \( n \times m \) matrix \( A \) to get \( xA \), an \( m \)-dimensional row vector). In what follows, we consider \( n \)-dimensional real vector space \( E \) and its dual \( E^* \). For the sake of illustration, we take the elements of \( E \) as column vectors and the elements of \( E^* \) as row vectors.
Definition 0.14. A \((p, q)\)-tensor associated to a vector space \(E\) of dimension \(n\) is a tensor of \(E_1 \otimes E_2 \otimes \cdots \otimes E_{p+q}\) where \(E_i = E\) for \(q\) values of \(i\) and \(E_j = E^*\), the dual space of \(E\), for \(p\) values of \(j\). The tensor is said to be \(p\) times covariant and \(q\) times contravariant. We denote the set of \((p, q)\) tensor attached to \(E\) as \(\bigotimes^p E^* \otimes E\).