0.2. Tensors

Note. In this section we define the tensor product of two finite dimensional vector spaces, tensor, covariant, contrapositive, and illustrate these ideas with examples.

Definition 0.13. Let E and F be two real vector spaces of dimensions n and p, respectively. The *tensor product* of E and F is a vector space of dimension np, denoted $E \otimes F$. A vector of $E \otimes F$ is called a *tensor*. For $x \in E$ and $y \in F$ we we associate $x \otimes y \in E \otimes F$. This product has the following properties:

(a) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$ and $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$ where $x, x_1, x_2 \in E$ and $y, y_1, y_2 \in F$.

(b) If $\alpha \in \mathbb{R}$ is a scalar then

$$(\alpha x) \otimes y = x \otimes (\alpha y) = \alpha(x \otimes y).$$

(c) If $\{e_i\}_{1 \le i \le n}$ is a basis of E and $\{f_j\}_{1 \le j \le p}$ is a basis of F, then $e_i \otimes f_j$ is a basis of $E \otimes F$.

(d) If G is a third real vector space with $E \otimes (F \otimes G) = (E \otimes F) \otimes G$, then

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

for all $z \in G$.

Note. By part (d) of the definition, the product of tensors is an associative operation. However, we have not assumed commutivity.

Note/Definition. We follow the *Einstein convention* when dealing with summations. When the index (such as i or j) is present above and below (that is, when the index is present as both a superscript and a subscript) then summation over the index is assumed. In $E \otimes F$ where where E is dimension n and F is dimension p, i is summed from 1 to n and j is summed from 1 to p.

Example. Let $x \in E$ and $y \in F$ where

$$x = \sum_{i=1}^{n} x^i e_i = x^i e_i \text{ and } y = \sum_{j=1}^{p} y^j f_j = y^j f_j.$$

By (a) in the definition of product of tensors,

$$x \otimes y = (x^i e_i) \otimes (y^j f_j) = \left(\sum_{i=1}^{n} x^i e_i \right) \otimes \left(\sum_{j=1}^{p} y^j f_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{p} (x^i y^j)(e_i \otimes f_j) = x^i y^j \, e_i \otimes f_j.$$

So the components of $x \otimes y$ with respect to the basis $\{e_i \otimes f_j\}$ (for $1 \le i \le n$ and $1 \le j \le p$) of $E \otimes F$ are $t^{ij} = x^i y^j$.

Note. Suppose that both $\{e_i\}_{1 \le i \le n}$ and $\{\tilde{e}_\alpha\}_{1 \le \alpha \le n}$ are bases for E and that both $\{f_j\}_{1 \le j \le p}$ and $\{\tilde{b}_\beta\}_{1 \le \beta \le p}$ are bases for F. Then we can write

$$\tilde{e}_\alpha = \sum_{i=1}^{n} a^i_\alpha e_i = a^i_\alpha e_i \text{ and } \tilde{f}_\beta = \sum_{\beta=1}^{p} c^j_\beta f_j = c^j_\beta f_j,$$

or

$$e_i = \sum_{\alpha=1}^{n} b^\alpha_i \tilde{e}_\alpha = b^\alpha_i \tilde{e}_\alpha \text{ and } f_j = \sum_{\beta=1}^{p} d^\beta_j \tilde{f}_\beta = d^\beta_j \tilde{f}_\beta.$$

Here, the matrices (a^i_α) (with $1 \leq i, \alpha \leq n$) and (b^α_i) (with $1 \leq i, \alpha \leq n$) are inverses of each other, and matrices (c^j_β) (with $1 \leq j, \beta \leq p$) and (d^β_j) (with $1 \leq j, \beta \leq p$) are inverses of each other:

$$(b^\alpha_i)(a^i_\alpha) = \begin{pmatrix} b^1_1 & b^2_1 & \cdots & b^n_1 \\ b^1_2 & b^2_2 & \cdots & b^n_2 \\ \vdots & \vdots & \ddots & \vdots \\ b^1_n & b^2_n & \cdots & b^n_n \end{pmatrix} \begin{pmatrix} a^1_1 & a^2_1 & \cdots & a^n_1 \\ a^1_2 & a^2_2 & \cdots & a^n_2 \\ \vdots & \vdots & \ddots & \vdots \\ a^1_n & a^2_n & \cdots & a^n_n \end{pmatrix}$$

$$= \begin{pmatrix} b^\alpha_1 a^1_\alpha & b^\alpha_1 a^2_\alpha & \cdots & b^\alpha_1 a^n_\alpha \\ b^\alpha_2 a^1_\alpha & b^\alpha_2 a^2_\alpha & \cdots & b^\alpha_2 a^n_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ b^\alpha_n a^1_\alpha & b^\alpha_n a^2_\alpha & \cdots & b^\alpha_n a^n_\alpha \end{pmatrix} = (b^\alpha_i a^j_\alpha)$$

where $e_i = b^\alpha_i \tilde{e}_\alpha = b^\alpha_i (a^j_\alpha e_j) = b^\alpha_i a^j_\alpha e_j$ so that $b^\alpha_i a^j_\alpha = \delta^j_i = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}$ and hence $(b^\alpha_i)(a^i_\alpha) = \mathcal{I}$. Here, δ^j_i is called the *Kronecker tensor*. Similarly, $(a^j_\alpha)(b)i^\alpha) = \mathcal{I}$ (or this follows since the matrices are square; see "Theorem 1.11. A Commutative Property" in my online notes for 1.5. Inverses of Square Matrices). So for tensor $T = t^{\alpha\beta}\, \tilde{e}_\alpha \otimes \tilde{f}_\beta = t^{ij}\, e_i \otimes f_j$ we have

$$T = t^{ij}\, e_i \otimes f_j = t^{ij}(b^\alpha_i \tilde{e}_\alpha) \otimes (d^\beta_j \tilde{f}_\beta) = t^{ij}\, b^\alpha_i d^\beta_j\, \tilde{e}_\alpha \otimes \tilde{f}_\beta = t^{\alpha\beta}\, \tilde{a}_\alpha \otimes \tilde{f}_\beta$$

so that we can relate the coordinates of T with respect to the different bases as $t^{\alpha\beta} = t^{ij} b^\alpha_i d^\beta_j$ and similarly $t^{ij} = t^{\alpha\beta} a^i_\alpha c^j_\beta$.

Note/Definition. Recall that the collection of linear mappings between (finite dimensional real) vector spaces X and Y form a vector space themselves, denoted $L(X, Y)$ (see page 24 of C.T.J Dodson and T. Poston's *Tensor Geometry: The Geometric Viewpoint and its Uses*, 2nd Edition, Graduate Texts in Mathematics #130, Springer Verlag (1991)). In fact, if X is dimension m and Y is dimension n, so that $X \cong \mathbb{R}^n$ and $Y \cong \mathbb{R}^m$ (by the Fundamental Theorem of Finite Dimensional Vector Spaces; see my online notes for 3.3 Coordinatization of Vectors, page 5), then each element of $L(X, Y)$ is represented by a $m \times n$ matrix (see my online notes for 3.4. Linear Transformations, "Theorem 3.10. Matrix Representations of Linear Transformations" on page 10). Here, we interpret the vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ as column vectors. In the event that $Y = \mathbb{R}$, an element of $L(X, \mathbb{R})$ ix called a *linear functional* (or *dual vector* or *covariant vector*). The vector space $L(X, \mathbb{R})$ is the *dual space* of X, denoted X^* (see page 57 of *Tensor Geometry*). With $X = \mathbb{R}^n$, the vector space of n-dimensional column vectors, the dual space is X^* the vector space of n-dimensional row vectors. We then get the functional action produced by multiplying $n \times 1$ vector $x \in X$ on the left by some $1 \times n$ vector $y \in X^*$, producing effectively a dot product. We should not that above, $m \times n$ matrix A is applied to n-dimensional column vector x to produce m-dimensional column vector $Ax \in Y$ (otherwise we take row vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ and an $n \times m$ matrix A to get xA, an m-dimensional row vector). In what follows, we consider n-dimensional real vector space E and its dual E^*. For the sake of illustration, we take the elements of E as column vectors and the elements of E^* as row vectors.

Definition 0.14. A (p, q)-*tensor* associated to a vector space E of dimension n is a tensor of $E_1 \otimes E_2 \otimes \cdots \otimes E_{p+q}$ where $E_i = E$ for q values of i and $E_j = E^*$, the dual space of E, for p values of j. The tensor is said to be p *times covariant and* q *times contravariant*. We denote the set of (p, q) tensor attached to E as $\overset{p}{\otimes} E^* \overset{q}{\otimes} E$.

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