1.3. The Spacetime Metric

Note. The Lorentz transformation of Section 1.2 shows that different observers do not agree on spatial and temporal measurements. If we use a delta ("Δ") to represent changes in a quantity then we get the following relations from the Lorentz transformation:

$$\Delta t' = \frac{\Delta t - v\Delta x/c}{\sqrt{1 - v^2/c^2}}, \quad \Delta x' = \frac{\Delta x - vt}{\sqrt{1 - v^2/c^2}}, \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z.$$

Definition. The (spacetime) *interval* between two events (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is

$$I = -(\Delta t)^2 + \frac{1}{c^2}\left((\Delta t)^2 + (\Delta y)^2 + (\Delta z)^2\right)$$

where $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, and $\Delta z = z_2 - z_1$.

Note. Let $p_1 = (t_1, x_1, y_1, z_1) = (t_1', x_1', y_1', z_1')$ and $p_2 = (t_2, x_2, y_2, z_2) = (t_2', x_2', y_2', z_2')$ be events where the coordinates (t, x, y, z) and (t', x', y', z') are related by the Lorentz transformation. Then the intervals I and I' satisfy:

$$\begin{aligned}
I' &= -(\Delta t')^2 + \frac{1}{c^2}\left((\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2\right) \\
&= -\left(\frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}\right)^2 + \frac{1}{c^2}\left(\frac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}}\right)^2 + \frac{(\Delta y)^2}{c^2} + \frac{(\Delta z)^2}{c^2} \\
&= \frac{-(\Delta t)^2 + 2v\Delta t\Delta x/c^2 - v^2(\Delta x)^2/c^4 + (\Delta x)^2/c^2 - 2v\Delta t\Delta x/c^2 + v^2(\Delta t)^2/c^2}{1 - v^2/c^2} \\
&\quad + \frac{(\Delta y)^2}{c^2} + \frac{(\Delta z)^2}{c^2} \\
&= \frac{-(\Delta t)^2(1 - v^2/c^2)}{1 - v^2/c^2} + \frac{1}{c^2}\frac{(\Delta x)^2(1 - v^2/c^2}{1 - v^2/c^2} + \frac{(\Delta y)^2}{c^2} + \frac{(\Delta z)^2}{c^2}
\end{aligned}$$

$$= -(\Delta t)^2 + \frac{1}{c^2}\left((\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right) = I.$$

That is, inertial observers disagree on the temporal and spatial separation between two events but they agree on the *interval* between two events. This reveals the ironic fact that special relativity deals with a quantity that is an absolute between inertial frames: The interval. The interval is the "spacetime metric" in the title of this section (though, strictly speaking, it is not a metric since metrics must be nonnegative-valued).

Note. We can now see how there is a relativity of simultaneity. That is, if for events p_1 and p_2 (with the notation above), we may have $\Delta t = 0$ (and simultaneity of the events in (t, x, y, z) coordinates) but $\Delta t' \neq 0$ (and nonsimultaneity in (t', x', y', z') coordinates). Since $I = I'$, we would need $(\Delta x)^2 = -(\Delta t')^2 + (\Delta x')^2$. This is amusingly illustrated in Problem 1.1.

Revised: 6/19/2018