Chapter 2: Manifolds and Tensor Fields

2.1. Manifolds

Note. An event is a point in spacetime. In prerelativity physics and in special relativity, the space of all events is $\mathbb{R}^4$. In general relativity, we will keep the idea that spacetime is locally "like" $\mathbb{R}^4$, but allow spacetime to have geometric properties different from $\mathbb{R}^4$ (that is, we do not require spacetime to be flat). This is analogous to the fact that the Earth is locally like $\mathbb{R}^2$, but is globally different from $\mathbb{R}^2$.

Note. When making an analogy with the surface of the Earth, we take advantage of the fact that a sphere is embedded in 3-space (an extrinsic property). When considering all of spacetime, it would not make sense to think of how it is embedded in a higher dimensional space (for this would imply something “outside” of the universe, and such things are beyond science — if not beyond meaning!). We must therefore study spacetime from within (that is, we can only study intrinsic properties of spacetime). For such a study, we need to develop the abstract idea of an $n$-manifold.
2.1. Manifolds

**Definition.** Let \( x = (x^1, x^2, \ldots, x^n) \) and \( y = (y^1, y^2, \ldots, y^n) \) be points in \( \mathbb{R}^n \). Define the (Euclidean) distance between \( x \) and \( y \) as

\[
|x - y| = \left[ \sum_{\mu=1}^{n} (x^\mu - y^\mu)^2 \right]^{1/2}.
\]

The **open ball** in \( \mathbb{R}^n \) of center \( y \) and radius \( r \) is

\[
\{ x \in \mathbb{R}^n \mid |x - y| < r \}.
\]

An **open set** in \( \mathbb{R}^n \) is any set which can be expressed as an arbitrary union of open balls.

**Note.** In fact, an open set in \( \mathbb{R}^n \) can be expressed as a countable collection of open balls. This is called the **Lindelöf Property** of \( \mathbb{R}^n \).

**Definition.** A function \( \varphi : X \to Y \) is **one-to-one** if for distinct \( x^1, x^2 \in X \) we have \( \varphi(x^1) \neq \varphi(x^2) \). (If a function \( \varphi \) is one-to-one, then we can define its inverse \( \varphi^{-1} \).) The function \( \varphi \) is **onto** if for each \( y \in Y \), there exists \( x \in X \) such that \( \varphi(x) = y \). If \( \varphi : X \to \mathbb{R}^n \) (where \( X \) is any set) then \( \varphi \) is \( C^\infty \) is \( \varphi \) is infinitely differentiable. (Notice that \( \varphi \) is a vector valued function and should be treated as an \( n \)-tuple of scalar valued functions \( \varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n) \). Saying \( \varphi \) is infinitely differentiable is equivalent to saying that \( \varphi^\mu \) is infinitely differentiable for each \( \mu \). It does not make sense to talk about the differentiability of a function between arbitrary sets — we need more structure than that.)
Note. In the opinion of your humble instructor, a manifold is best thought of in terms of paper mache. As opposed to taking small pieces of paper (which represent little open sets from a 2-dimensional vector space) and soaking them in water and glue (allowing us to bend and warp the pieces; that is, to continuously transform them), we consider small pieces of \( n \)-dimensional space which are mapped continuously to open subsets of the manifold. Instead of pasting the pieces of paper onto a wire frame and overlapping them, we require that the continuous mappings compose to give a certain level of differentiability. More precisely, we have the following.

**Definition.** An \( n \)-dimensional, \( C^\infty \), real manifold is a set of \( M \) points together with a collection \( \{O_\alpha\} \) of subsets of \( M \) such that

1. \( \bigcup_\alpha O_\alpha = M \).

2. For each \( \alpha \), there is a one-to-one and onto map \( \psi_\alpha : O_\alpha \to U_\alpha \) where \( U_\alpha \) is an open subset of \( \mathbb{R}^n \).

3. For any two \( O_\alpha, O_\beta \subset M \) with \( O_\alpha \cap O_\beta \neq \emptyset \). We have the sets \( \psi_\alpha[O_\alpha \cap O_\beta] \subset \mathbb{R}^n \) and \( \psi_\beta[O_\alpha \cap O_\beta] \subset \mathbb{R}^n \) open and the function \( \psi_\beta \circ \psi_\alpha^{-1} \) is \( C^\infty \).

Notice that \( \psi_\beta \circ \psi_\alpha^{-1} \) maps \( \mathbb{R}^n \) to \( \mathbb{R}^n \), so differentiability is defined. Each map \( \psi_\alpha \) is a coordinate system (as physicists say; mathematicians call it a “chart”).

**Convention.** For a manifold \( M \), we require that the cover \( \{O_\alpha\} \) and the set of coordinate systems \( \{\psi_\alpha\} \) are maximal. That is, all coordinate systems compatible with parts 2 and 3 of the definition of a manifold are included. This avoids the complication of defining new manifolds from given manifolds by simply adding or deleting a coordinate system.
Note. We say $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^\infty$ if all partials (pure and mixed) of all orders exist and are continuous.

Definition. The maps $\psi_\alpha$ above which associate elements of $\mathbb{R}^n$ with points of the manifold are called charts or coordinate systems.

Note. The function $\psi_\beta \circ \psi^{-1}_\alpha$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\psi_\alpha[O_\alpha \cap O_\beta] \subset U_\alpha$ and $\psi_\beta[O_\alpha \cap O_\beta] \subset U_\beta$. We have:

Example. $\mathbb{R}^n$ is an $n$-manifold (trivially).
Example. (Problem 2.1(a)) The 2-sphere

\[ S^2 = \{(x^1, x^2, x^3) \in \mathbb{R} | (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\} \]

is a 2-manifold. First, we cannot simply map \( S^2 \) continuously onto \( \mathbb{R}^2 \) since \( S^2 \) is a compact subset of \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \) is not a compact set of \( \mathbb{R}^3 \) (and continuous functions map compact sets onto compact sets). Therefore we have to cover \( S^2 \) with sets \( \{O_\alpha\} \) and map these sets into \( \mathbb{R}^2 \). So define the six hemispherical open sets \( O_i^\pm \) for \( i = 1, 2, 3 \):

\[ O_i^\pm = \{(x^1, x^2, x^3) \in S^2 | \pm x^i > 0\} \]

these correspond to the top half, bottom half, right half, left half, front half, and bottom half of the sphere). Then Property 1 of the definition of manifold is satisfied: \( \bigcup O_\alpha = S^2 = M \). Next, take \( U_\alpha = D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\} \) for \( \alpha = 1, 2, \ldots, 6 \). We define the six \( \psi_\alpha \) by projecting the hemisphere onto \( D \):

Each of the six (open) hemispheres are mapped onto the (open) unit disk.
That is, \( f^+_1(x^1, x^2, x^3) = (x^2, x^3), f^+_2(x^1, x^2, x^3) = (x^1, x^3), \) and \( f^+_3(x^1, x^2, x^3) = (x^1, x^2). \) Then the coordinate system \( \{\psi_\alpha\} = \{f^+_i\} \) are one-to-one and onto maps mapping the \( O_\alpha \)'s onto open set \( D. \) Hence property 2 of the definition of manifold is satisfied. (The following is Problem 2.1a.) Now \((f_j^\pm)^{-1}\) maps \( D \) onto one of the six hemispheres. So \((f_i^\pm) \circ (f_j^\pm)^{-1}\) maps \( D \) onto a (not necessarily nonempty) subset of \( D. \) In particular, \((f_i^+) \circ (f_i^-)^{-1}\) and \((f_i^-) \circ (f_i^+)^{-1},\) for \( i = 1, 2, 3 (so this covers 6 cases) \) are nowhere defined and Property 3 does not apply. (For example, \((f_3^+)\)^{-1}\) maps \( D \) onto the upper (open) hemisphere of \( S^2 \) but \( f^-_3 \) is only defined on the lower [open] hemisphere of \( S^2 \) so \((f^-_3) \circ (f^+_3)^{-1}\) has an empty domain.) Next, if \( i \neq j \) then \((f_i^\pm) \circ (f_j^\pm)^{-1}\) maps \( D \) onto one of the following (depending on \( i \) and \( j\)):


Without loss of generality, consider \((f_2^+) \circ (f_1^+)^{-1}. \) Then \((f_1^+)^{-1}\) maps \( D \) onto the “front” of \( S^2 \) and \( f_2^+ \) then maps the intersection of the “front” and “right” of \( S^2 \) onto the right side of \( D. \)
So
\[
(f_1^+)^{-1}(x, y) \longrightarrow (\sqrt{1 - x^2 - y^2}, x, y) \quad \text{and} \quad f_2^+(x, y) \longrightarrow (\sqrt{1 - x^2 - y^2}, y).
\]
For \((x, y) \in D\), this composition is \(C^\infty\) (notice that the + need not correspond to the +, and so this covers \(4 \times 6 = 24\) cases). This leaves 6 of the \(6 \times 6 = 36\) cases to consider, namely \((f_i^+) \circ (f_i^+)^{-1}\) and \((f_i^-) \circ (f_i^-)^{-1}\) for \(i = 1, 2, 3\). In each of these 6 cases, the composition is the identity function from \(D\) to \(D\) and so is \(C^\infty\). Therefore, \textit{Property 3 of the definition of manifold is satisfied} and hence \(S^2\) is a manifold. □

\textbf{Note.} Analogous to the previous example, we can show that the \(n\)-sphere \(S^n\) is an \(n\)-manifold.

\textbf{Definition.} Given two manifolds \(M\) and \(M'\) of dimensions \(n\) and \(n'\), respectively, we define the \textit{product manifold} \(M \times M'\). We take as the points of \(M \times M'\) the collection of all pairs \((p, p')\) where \(p \in M\) and \(p' \in M'\). For the collection of subsets of \(M \times M'\) we take the collection of all \(\{O_{\alpha\beta} = O_\alpha \times O_\beta\}\) where \(O_\alpha\) is a subset of \(M\) and \(O_\beta\) is a subset of \(M'\) (where \(O_\alpha\) and \(O_\beta\) are as described in the definition of manifold). Finally, we define the coordinate system \(\{\varphi_{\alpha\beta}\}\) where \(\varphi_{\alpha\beta} : O_{\alpha\beta} \rightarrow U_{\alpha\beta} = U_\alpha \times U_\beta \subset \mathbb{R}^{n+n'}\) as \(\psi_{\alpha\beta}(p, p') = (\psi_\alpha(p), \psi_\beta(p'))\) where \(p \in O_\alpha\), \(p' \in O_\beta\), \(\psi_\alpha : O_\alpha \rightarrow U_\alpha\), and \(\psi_\beta : O_\beta \rightarrow U'_\beta\).

\textbf{Note.} Certainly \(\{O_{\alpha\beta}\}\) above covers \(M \times M'\). Also, for all \(\alpha\beta\), the function \(\psi_{\alpha\beta}\) maps \(O_{\alpha\beta}\) one-to-one and onto an open set of \(\mathbb{R}^{n+n'}\) (namely, \(U_{\alpha\beta} = U_\alpha \times U'_\beta\)).
Note. We now can define differentiability of a function from one manifold $M$ to another $M'$. We do so by using the charts of the manifolds. Let $M$ and $M'$ have chart maps $\{\psi_\alpha\}$ and $\{\psi'_\beta\}$, respectively, and let $f : M \to M'$. Then we have

$$\mathbb{R}^n \xrightarrow{\psi^{-1}_\alpha} M \xrightarrow{f} M' \xrightarrow{\psi'_\beta} \mathbb{R}^{n'},$$

and $\psi'_\beta \circ f \circ \psi^{-1}_\alpha : \mathbb{R}^n \to \mathbb{R}^{n'}$. So we use this mapping to define differentiability of $f$.

**Definition.** Let $M$ and $M'$ be manifolds with chart maps $\{\psi_\alpha\}$ and $\{\psi'_\beta\}$, respectively. We have $f : M \to M'$ is $C^\infty$ if for each $\alpha$ and $\beta$ the map $\psi'_\beta \circ f \circ \psi^{-1}_\alpha$ taking $U_\alpha \subset \mathbb{R}^n$ into $U'_\beta \subset \mathbb{R}^{n'}$ is $C^\infty$. If $f : M \to M'$ is $C^\infty$, one to one, onto and has a $C^\infty$ inverse, then $f$ is a **diffeomorphism** and $M$ and $M'$ are said to be **diffeomorphic**.

Note. Diffeomorphic manifolds have identical manifold structure.

Note. These notes are based in part on the fall 2011 Honors in Discipline project of Jessie Deering-Jamieson titled “What is a Manifold?”

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