Chapter 2: Manifolds and Tensor Fields

2.1. Manifolds

Note. An *event* is a point in spacetime. In prerelativity physics and in special relativity, the space of all events is $\mathbb{R}^4$. In general relativity, we will keep the idea that spacetime is locally "like" $\mathbb{R}^4$, but allow spacetime to have geometric properties different from $\mathbb{R}^4$ (that is, we do not require spacetime to be flat). This is analogous to the fact that the Earth is locally like $\mathbb{R}^2$, but is globally different from $\mathbb{R}^2$.

Note. When making an analogy with the surface of the Earth, we take advantage of the fact that a sphere is embedded in 3-space (an *extrinsic* property). When considering all of spacetime, it would not make sense to think of how it is embedded in a higher dimensional space (for this would imply something "outside" of the universe, and such things are beyond science — if not beyond meaning!). We must therefore study spacetime from within (that is, we can only study *intrinsic* properties of spacetime). For such a study, we need to develop the abstract idea of an n-manifold.

Definition. Let $x = (x^1, x^2, \ldots, x^n)$ and $y = (y^1, y^2, \ldots, y^n)$ be points in $\mathbb{R}^n$. Define the (Euclidean) *distance* between x and y as

$$|x - y| = \left[\sum_{\mu=1}^{n} (x^\mu - y^\mu)^2 \right]^{1/2}.$$

The *open ball* in $\mathbb{R}^n$ of center y and radius r is

$$\{x \in \mathbb{R}^n \mid |x - y| < r\}.$$

An *open set* in $\mathbb{R}^n$ is any set which can be expressed as an arbitrary union of open balls.

Note. In fact, an open set in $\mathbb{R}^n$ can be expressed as a countable collection of open balls. This is called the *Lindelöf Property* of $\mathbb{R}^n$.

Definition. A function $\varphi : X \to Y$ is *one-to-one* if for distinct $x^1, x^2 \in X$ we have $\varphi(x^1) \neq \varphi(x^2)$. (If a function φ is one-to-one, then we can define its inverse φ^{-1}.) The function φ is *onto* if for each $y \in Y$, there exists $x \in X$ such that $\varphi(x) = y$. If $\varphi : X \to \mathbb{R}^n$ (where X is any set) then φ is C^∞ is φ is infinitely differentiable. (Notice that φ is a vector valued function and should be treated as an n-tuple of scalar valued functions $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n)$. Saying φ is infinitely differentiable is equivalent to saying that φ^μ is infinitely differentiable for each μ. It does not make sense to talk about the differentiability of a function between arbitrary *sets* — we need more structure than that.)

Note. In the opinion of your humble instructor, a manifold is best thought of in terms of paper mache. As opposed to taking small pieces of paper (which represent little open sets from a 2-dimensional vector space) and soaking them in water and glue (allowing us to bend and warp the pieces; that is, to continuously transform them), we consider small pieces of n-dimensional space which are mapped continuously to open subsets of the manifold. Instead of pasting the pieces of paper onto a wire frame and overlapping them, we require that the continuous mappings compose to give a certain level of differentiability. More precisely, we have the following.

Definition. An *n-dimensional, C^∞, real manifold* is a set of M *points* together with a collection $\{O_\alpha\}$ of subsets of M such that

1. $\bigcup_\alpha O_\alpha = M$.

2. For each α, there is a one-to-one and onto map $\psi_\alpha : O_\alpha \to U_\alpha$ where U_α is an open subset of $\mathbb{R}^n$.

3. For any two $O_\alpha, O_\beta \subset M$ with $O_\alpha \bigcap O_\beta \neq \emptyset$. We have the sets $\psi_\alpha[O_\alpha \bigcap O_\beta] \subset \mathbb{R}^n$ and $\psi_\beta[O_\alpha \bigcap O_\beta] \subset \mathbb{R}^n$ open and the function $\psi_\beta \circ \psi_\alpha^{-1}$ is C^∞.

Notice that $\psi_\beta \circ \psi_\alpha^{-1}$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$, so differentiability is defined. Each map ψ_α is a *coordinate system* (as physicists say; mathematicians call it a "chart").

Convention. For a manifold M, we require that the cover $\{O_\alpha\}$ and the set of coordinate systems $\{\psi_\alpha\}$ are maximal. That is, all coordinate systems compatible with parts 2 and 3 of the definition of a manifold are included. This avoids the complication of defining new manifolds from given manifolds by simply adding or deleting a coordinate system.

Note. We say $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ if all partials (pure and mixed) of all orders exist and are continuous.

Definition. The maps ψ_α above which associate elements of $\mathbb{R}^n$ with points of the manifold are called *charts* or *coordinate systems*.

Note. The function $\psi_\beta \circ \psi_\alpha^{-1}$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\psi_\alpha[O_\alpha \bigcap O_\beta] \subset U_\alpha$ and $\psi_\beta[O_\alpha \bigcap O_\beta] \subset U_\beta$. We have:

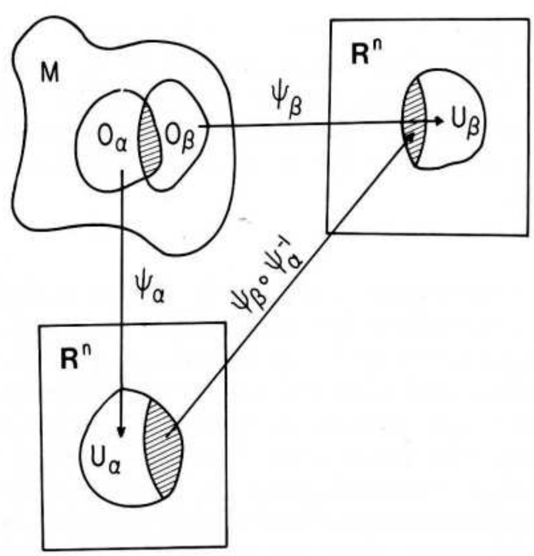


Figure 2.1 from Wald, page 13.

Example. $\mathbb{R}^n$ is an n-manifold (trivially).

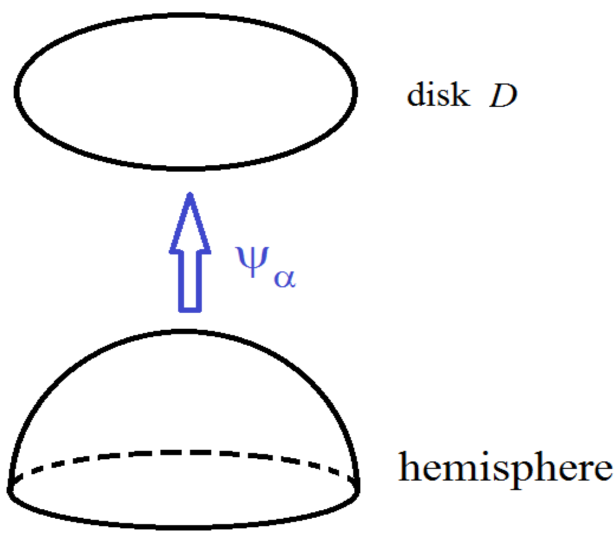
Example. (Problem 2.1(a)) The 2-sphere

$$S^2 = \{(x^1, x^2, x^3) \in \mathbb{R} \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

is a 2-manifold. First, we cannot simply map S^2 continuously onto $\mathbb{R}^2$ since S^2 is a compact subset of $\mathbb{R}^3$ and $\mathbb{R}^2$ is not a compact set of $\mathbb{R}^3$ (and continuous functions map compact sets onto compact sets). Therefore we have to cover S^2 with sets $\{O_\alpha\}$ and map these sets into $\mathbb{R}^2$. So define the six hemispherical open sets $O_i^\pm$ for $i = 1, 2, 3$:

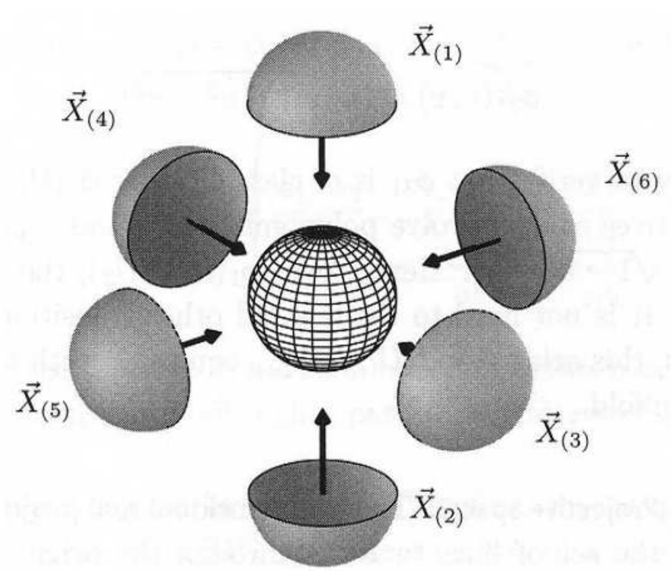
$$O_I^\pm = \{(x^1, x^2, x^3) \in S^2 \mid \pm x^i > 0\}$$

these correspond to the top half, bottom half, right half, left half, front half, and bottom half of the sphere). Then *Property 1 of the definition of manifold is satisfied*: $\bigcup_\alpha O_\alpha = S^2 = M$, Next, take $U_\alpha = D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ for $\alpha = 1, 2, \ldots, 6$. We define the six ψ_α by projecting the hemisphere onto D:



Each of the six (open) hemispheres are mapped onto the (open) unit disk.

That is, $f_1^\pm(x^1, x^2, x^3) = (x^2, x^3)$, $f_2^\pm(x^1, x^2, x^3) = (x^1, x^3)$, and $f_3^\pm(x^1, x^2, x^3) = (x^1, x^2)$. Then the coordinate system $\{\psi_\alpha\} = \{f_i^\pm\}$ are one-to-one and onto maps mapping the O_α's onto open set D. Hence *property 2 of the definition of manifold is satisfied.* (The following is Problem 2.1a.) Now $(f_j^\pm)^{-1}$ maps D onto one of the six hemispheres. So $(f_i^\pm) \circ (f_j^\pm)^{-1}$ maps D onto a (not necessarily nonempty) subset of D. In particular, $(f_i^+) \circ (f_i^-)^{-1}$ and $(f_i^-) \circ (f_i^+)^{-1}$, for $i = 1, 2, 3$ (so this covers 6 cases) are nowhere defined and Property 3 does not apply. (For example, $(f_3^+)^{-1}$ maps D onto the upper (open) hemisphere of S^2 but f_3^- is only defined on the lower [open] hemisphere of S^2 so $(f_3^-) \circ (f_3^+)^{-1}$ has an empty domain.) Next, if $i \neq j$ then $(f_i^\pm) \circ (f_j^\pm)^{-1}$ maps D onto one of the following (depending on i and j):



From *Differential Geometry of Manifolds*, by Stephen Lovett (A. K. Peters, 2010).

Without loss of generality, consider $(f_2^+) \circ (f_1^+)^{-1}$. Then $(f_1^+)^{-1}$ maps D onto the "front" of S^2 and f_2^+ then maps the intersection of the "front" and "right" of S^2 onto the right side of D.

So

$$(x,y) \overset{(f_1^+)^{-1}}{\longrightarrow} (\sqrt{1-x^2-y^2}, x, y) \overset{f_2^+}{\longrightarrow} (\sqrt{1-x^2-y^2}, y).$$

For $(x,y) \in D$, this composition is C^∞ (notice that the + need not correspond to the +, and so this covers $4 \times 6 = 24$ cases). This leaves 6 of the $6 \times 6 = 36$ cases to consider, namely $(f_i^+) \circ (f_i^+)^{-1}$ and $(f_i^-) \circ (f_i^-)^{-1}$ for $i = 1, 2, 3$. In each of these 6 cases, the composition is the identity function from D to D and so is C^∞. Therefore, *Property 3 of the definition of manifold is satisfied* and hence S^2 is a manifold. □

Note. Analogous to the previous example, we can show that the n-sphere S^n is an n-manifold.

Definition. Given two manifolds M and M' of dimensions n and n', respectively, we define the *product manifold* $M \times M'$. We take as the points of $M \times M'$ the collection of all pairs (p, p') where $p \in M$ and $p' \in M'$. For the collection of subsets of $M \times M'$ we take the collection of all $\{O_{\alpha\beta} = O_\alpha \times O_\beta\}$ where O_α is a subset of M and O_β is a subset of M' (where O_α and O_β are as described in the definition of manifold). Finally, we define the coordinate system $\{\varphi_{\alpha\beta}\}$ where $\varphi_{\alpha\beta} : O_{\alpha\beta} \to U_{\alpha\beta} = U_\alpha \times U_\beta \subset \mathbb{R}^{n+n'}$ as $\psi_{\alpha\beta}(p, p') = (\psi_\alpha(p), \psi'_\beta(p'))$ where $p \in O_\alpha$, $p' \in O_\beta$, $\psi_\alpha : O_\alpha \to U_\alpha$, and $\psi'_\beta : O'_\beta \to U'_\beta$.

Note. Certainly $\{O_{\alpha\beta}\}$ above covers $M \times M'$. Also, for all $\alpha\beta$, the function $\psi_{\alpha\beta}$ maps $O_{\alpha\beta}$ one-to-one and onto an open set of $\mathbb{R}^{n+n'}$ (namely, $U_{\alpha\beta} = U_\alpha \times U'_\beta$).

Note. We now can define differentiability of a function from one manifold M to another M'. We do so by using the charts of the manifolds. Let M and M' have chart maps $\{\psi_\alpha\}$ and $\{\psi'_\beta\}$, respectively, and let $f : M \to M'$. Then we have

$$\mathbb{R}^n \xrightarrow{\psi_\alpha^{-1}} M \xrightarrow{f} M' \xrightarrow{\psi'_\beta} \mathbb{R}^{n'},$$

and $\psi'_\beta \circ f \circ \psi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^{n'}$. So we use this mapping to define differentiability of f.

Definition. Let M and M' be manifolds with chart maps $\{\psi_\alpha\}$ and $\{\psi'_\beta\}$, respectively. We have $f : M \to M'$ is C^∞ if for each α and β the map $\psi'_\beta \circ f \circ \psi_\alpha^{-1}$ taking $U_\alpha \subset \mathbb{R}^n$ into $U'_\beta \subset \mathbb{R}^{n'}$ is C^∞. If $f : M \to M'$ is C^∞, one to one, onto and has a C^∞ inverse, then f is a *diffeomorphism* and M and M' are said to be *diffeomorphic.*

Note. Diffeomorphic manifolds have identical manifold structure.

Note. These notes are based in part on the fall 2011 Honors in Discipline project of Jessie Deering-Jamieson titled "What is a Manifold?"

Revised: 6/16/2018