# 2.2. Vectors (Partial)

**Note.** In this section, we introduce a way to discuss vectors tangent to a manifold intrinsically (that is, without an appeal to a "hyperspace" in which the manifold is embedded—curvature will have to be dealt with similarly).

**Note.** In Calculus III (MATH 2110), we used a vector (a *unit* vector) to define a directional derivative in $\mathbb{R}^2$ and $\mathbb{R}^3$. See Section 14.5 of my Calculus III notes (`http://faculty.etsu.edu/gardnerr/2110/notes-12e/c14s5.pdf`). For $(v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$ we have the directional derivative operator on $f(x^1, x^2, \ldots, x^n)$ defined as $\sum_{\mu=1}^{n} v^n \frac{\partial}{\partial x^\mu}[f]$ (and conversely, any directional derivative corresponds to a vector). Wald states (page 15) that "Directional derivatives are characterized by their linearity and 'Leibniz's Rule' [a version of the Product Rule] behavior when acting on functions." This motivates the following definition.

**Definition.** For manifold $M$, let $\mathcal{F}$ be the collection of all $C^\infty$ function from $M$ into $\mathbb{R}$. A *tangent vector* $v$ *at a point* $p \in M$ is a function $v : \mathcal{F} \to \mathbb{R}$ which satisfies:

**(1)** Linearity: $v(af + bg) = av(f) + bv(g)$ for all $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$, and

**(2)** Leibniz Rule: $v(fg) = f(p)v(g) + g(p)v(f)$.

**Note.** Notice that the only place the point $p$ plays a role in the definition of a tangent vector is in "Leibniz's Rule."

**Note.** If $h \in \mathcal{F}$ is a constant function, say $h(q) = c$ for all $q \in M$, then at point $p$ by Leibniz's Rule

$$v(h^2) = v(h\,h) = h(p)v(h) + h(p)v(h) = 2cv(h)$$

and by linearity

$$\begin{aligned} v(h^2) &= v(ch) \text{ since } h(q) = c \\ &= cv(h). \end{aligned}$$

So $v(h^2) = 2cv(h) = cv(h)$, and so $v(h) = 0$.

**Definition.** Let $V_p$ denote the collection of all tangent vectors at $p$ to manifold $M$. For $a, b \in \mathbb{R}$, define the *linear combination* $av_1 + bv_2 \in V_p$ as

$$(av_1 + bv_2)(f) = av_1(f) + bv_2(f)$$

for all $f \in \mathcal{F}$.

**Note.** $V_p$ is "clearly" a vector space (a vector space of linear operators on $\mathcal{F}$). The following result confirms that if manifold $M$ is of dimension $n$, then $V_p$ is of dimension $n$ as well.

**Theorem 2.2.1.** Let $M$ be an $n$-dimensional manifold. Let $p \in M$ and let $V_p$ denote the tangent space at $p$. Then $\dim(V_p) = n$.

**Definition.** The basis $\{X_\mu\}_{\mu=1}^n$ of $V_p$ (the $n$-dimensional tangent space to $M$ at $p$) of Theorem 2.2.1 is a *coordinate basis*.

**Note.** Notice that

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}\left[f \circ \psi^{-1}\right]\Big|_{\psi(p)}$$

where $f : M \to \mathbb{R}$ and $\psi : M \to \mathbb{R}^n$ so that $f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}$. Hence, basis vector $X_\mu$ depends on the coordinate system $\psi$. We could use a different coordinate system $\psi'$ to produce a different coordinate basis $\{X'_\nu\}$ at point $p$. We then want to relate the coordinate bases using the Chain Rule.

**Note.** By the Chain Rule of advanced Calculus (see, for example, my online Calculus 3 notes: `http://faculty.etsu.edu/gardnerr/2110/notes-12e/c14s4.pdf`) we have basis vector $X_\mu$ in terms of a different coordinate basis $\{X'_\nu\}$ as

$$X_\mu = \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu}\Big|_{\psi(p)} X'_\nu \qquad (2.2.9)$$

where $x'^\nu$ denotes the $\nu$th component of the map $\psi' \circ \psi^{-1}$. Equation (2.2.9) is to be established in Exercise 2.2.A.

**Note/Definition.** Since, as seen in the proof of Theorem 2.1.1, tangent vector $v$ is of the form $v(f) = \left(\sum_{\mu=1}^{n} v^\mu X_\mu\right)(f)$, or as an operator simply as

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu. \qquad (2.2.8)$$

Combining this with (2.2.9) gives the tangent vector in terms of basis $\{X'_\nu\}$ as

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu = \sum_{\mu=1}^{n} v^\mu \left( \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu}\Big|_{\psi(p)} X'_\nu \right)$$

$$= \sum_{\nu=1}^{n} \left( \sum_{\nu=1}^{n} v^{\mu} \frac{\partial x'^{\nu}}{\partial x^{\mu}} \bigg|_{\psi(p)} \right) X'_{\nu} = \sum_{\nu=1}^{n} v'^{\nu} X'_{\nu}.$$

So $v'^{\nu} = \sum_{\nu=1}^{n} v^{\mu} \frac{\partial x'^{\nu}}{\partial x^{\mu}}\big|_{\psi(p)}$ as point $p$, or at an operator

$$v'^{\nu} = \sum_{\nu=1}^{n} v^{\mu} \frac{\partial x'^{\nu}}{\partial x^{\mu}}.$$

This is called the *vector transformation law.*

**Definition.** A *smooth curve* $C$ on a manifold $M$ is a $C^{\infty}$ map of $\mathbb{R}$ (or an interval of $\mathbb{R}$ into $M$, $C : \mathbb{R} \to M$.

**Note.** For point $p \in M$ on smooth curve $C$ on $M$, we can associate a tangent vector $T \in V_p$ as follows. For $f \in \mathcal{F}$ (that is, for $f$ a $C^{\infty}$ function mapping $M$ to $\mathbb{R}$), set $T(f)$ equal to the derivative of $f \circ C : \mathbb{R} \to \mathbb{R}$ (here, $C : \mathbb{R} \to M$ and $f : M \to \mathbb{R}$) evaluated at $p$ (that is, evaluated at $t_0$ where $C(t_0) = p$):

$$T(f) = \frac{d}{dt}[f \circ C]\bigg|_{t=t_0}.$$

Notice that $T : \mathcal{F} \to \mathbb{R}$ and $T$ is linear. Also, for $f, g \in \mathcal{F}$ we have

$$T(fg) = \frac{d}{dt}[(fg) \circ C]\bigg|_{t=t_0} = \frac{d}{dt}[(f \circ C)(g \circ C)]\bigg|_{t=t_0}$$

$$= \frac{d}{dt}[f \circ C]\bigg|_{t=t_0} (g \circ C)\bigg|_{t=t_0} + (f \circ C)\bigg|_{t=t_0} \frac{d}{dt}[g \circ C]\bigg|_{t_0}$$

$$= T(f)g(C(t_0)) + f(C(t_0))T(g) = T(f)g(p) + f(p)T(g)$$

so that $T$ satisfies Leibniz Rule and hence by definition is a tangent vector to $M$ at point $p$.

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