2.2. Vectors

Note. In this section, we introduce a way to discuss vectors tangent to a manifold intrinsically (that is, without an appeal to a “hyperspace” in which the manifold is embedded—curvature will have to be dealt with similarly).

Note. In Calculus III (MATH 2110), we used a vector (a unit vector) to define a directional derivative in $\mathbb{R}^2$ and $\mathbb{R}^3$. See Section 14.5 of my Calculus III notes (http://faculty.etsu.edu/gardnerr/2110/notes-12e/c14s5.pdf). For $(v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$ we have the directional derivative operator on $f(x^1, x^2, \ldots, x^n)$ defined as $\sum_{\mu=1}^{n} v^n \frac{\partial}{\partial x^n} [f]$ (and conversely, any directional derivative corresponds to a vector). Wald states (page 15) that “Directional derivatives are characterized by their linearity and ‘Leibniz’s Rule’ [a version of the Product Rule] behavior when acting on functions.” This motivates the following definition.

Definition. For manifold $M$, let $\mathcal{F}$ be the collection of all $C^\infty$ function from $M$ into $\mathbb{R}$. A tangent vector $v$ at a point $p \in M$ is a function $v : \mathcal{F} \rightarrow \mathbb{R}$ which satisfies:

(1) Linearity: $v(af + bg) = av(f) + bv(g)$ for all $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$, and

(2) Leibniz Rule: $v(fg) = f(p)v(g) + g(p)v(f)$.

Note. Notice that the only place the point $p$ plays a role in the definition of a tangent vector is in “Leibniz’s Rule.”
Note. If $h \in \mathcal{F}$ is a constant function, say $h(q) = c$ for all $q \in M$, then at point $p$ by Leibniz’s Rule

$$v(h^2) = v(hh) = h(p)v(h) + h(p)v(h) = 2cv(h)$$

and by linearity

$$v(h^2) = v(ch) \text{ since } h(q) = c$$

$$= cv(h).$$

So $v(h^2) = 2cv(h) = cv(h)$, and so $v(h) = 0$.

Definition. Let $V_p$ denote the collection of all tangent vectors at $p$ to manifold $M$. For $a, b \in \mathbb{R}$, define the linear combination $av_1 + bv_2 \in V_p$ as

$$(av_1 + bv_2)(f) = av_1(f) + bv_2(f)$$

for all $f \in \mathcal{F}$.

Note. $V_p$ is “clearly” a vector space (a vector space of linear operators on $\mathcal{F}$). The following result confirms that if manifold $M$ is of dimension $n$, then $V_p$ is of dimension $n$ as well.

Theorem 2.2.1. Let $M$ be an $n$-dimensional manifold. Let $p \in M$ and let $V_p$ denote the tangent space at $p$. Then $\dim(V_p) = n$.

Definition. The basis $\{X_\mu\}_{\mu=1}^n$ of $V_p$ (the $n$-dimensional tangent space to $M$ at $p$) of Theorem 2.2.1 is a coordinate basis.
Note. Notice that

\[ X_\mu(f) = \frac{\partial}{\partial x^\mu} \left[ f \circ \psi^{-1} \right] \bigg|_{\psi(p)} \]

where \( f : M \to \mathbb{R} \) and \( \psi : M \to \mathbb{R}^n \) so that \( f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R} \). Hence, basis vector \( X_\mu \) depends on the coordinate system \( \psi \). We could use a different coordinate system \( \psi' \) to produce a different coordinate basis \( \{ X'_\nu \} \) at point \( p \). We then want to relate the coordinate bases using the Chain Rule.

Note. By the Chain Rule of advanced Calculus (see, for example, my online Calculus 3 notes: http://faculty.etsu.edu/gardnerr/2110/notes-12e/c14s4.pdf) we have basis vector \( X_\mu \) in terms of a different coordinate basis \( \{ X'_\nu \} \) as

\[ X_\mu = \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu} \bigg|_{\psi(p)} X'_\nu \tag{2.2.9} \]

where \( x'^\nu \) denotes the \( \nu \)-th component of the map \( \psi' \circ \psi^{-1} \). Equation (2.2.9) is to be established in Exercise 2.2.A.

Note/Definition. Since, as seen in the proof of Theorem 2.1.1, tangent vector \( v \) is of the form \( v(f) = \left( \sum_{\mu=1}^{n} v^\mu X_\mu \right) (f) \), or as an operator simply as

\[ v = \sum_{\mu=1}^{n} v^\mu X_\mu. \tag{2.2.8} \]

Combining this with (2.2.9) gives the tangent vector in terms of basis \( \{ X'_\nu \} \) as

\[ v = \sum_{\mu=1}^{n} v^\mu X_\mu = \sum_{\mu=1}^{n} v^\mu \left( \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu} \bigg|_{\psi(p)} X'_\nu \right) \]
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\[ 
= \sum_{\nu=1}^{n} \left( \sum_{\nu=1}^{n} v^\nu \frac{\partial x'\nu}{\partial x\mu} \right) \psi(p) X'_\nu = \sum_{\nu=1}^{n} v'\nu X'_\nu .
\]

So \( v'\nu = \sum_{\nu=1}^{n} v^\mu \frac{\partial x'\nu}{\partial x\mu} \) as point \( p \), or at an operator

\[ 
v'\nu = \sum_{\nu=1}^{n} v^\mu \frac{\partial x'\nu}{\partial x\mu} .
\]

This is called the vector transformation law.

**Definition.** A smooth curve \( C \) on a manifold \( M \) is a \( \mathcal{C}^\infty \) map of \( \mathbb{R} \) (or an interval of \( \mathbb{R} \) into \( M \), \( C : \mathbb{R} \to M \).

**Note.** For point \( p \in M \) on smooth curve \( C \) on \( M \), we can associate a tangent vector \( T \in V_p \) as follows. For \( f \in \mathcal{F} \) (that is, for \( f \) a \( \mathcal{C}^\infty \) function mapping \( M \) to \( \mathbb{R} \)), set \( T(f) \) equal to the derivative of \( f \circ C : \mathbb{R} \to \mathbb{R} \) (here, \( C : \mathbb{R} \to M \) and \( f : M \to \mathbb{R} \)) evaluated at \( p \) (that is, evaluated at \( t_0 \) where \( C(t_0) = p \)):

\[ T(f) = \left. \frac{d}{dt} [f \circ C] \right|_{t=t_0} .
\]

Notice that \( T : \mathcal{F} \to \mathbb{R} \) and \( T \) is linear. Also, for \( f, g \in \mathcal{F} \) we have

\[ T(fg) = \left. \frac{d}{dt} [(fg) \circ C] \right|_{t=t_0} = \left. \frac{d}{dt} [f \circ (g \circ C)] \right|_{t=t_0} = \left. \frac{d}{dt} [g \circ C] \right|_{t=t_0} \left. (g \circ C) \right|_{t=t_0} + \left. (f \circ C) \right|_{t=t_0} \left. \frac{d}{dt} [g \circ C] \right|_{t=t_0} = T(f)g(C(t_0)) + f(C(t_0))T(g) = T(f)g(p) + f(p)T(g)
\]

so that \( T \) satisfies Leibniz Rule and hence by definition is a tangent vector to \( M \) at point \( p \).