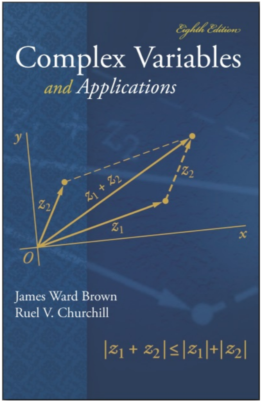
Complex Variables

Chapter 1. Complex Numbers

Section 1.11. Regions in the Complex Plane—Proofs of Theorems



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Lemma 1.11.A

Lemma 1.11.A. A point z_0 is a boundary point of set S if and only if every ε neighborhood of z_0 contains at least one point in set S and at least one point not in S.

Proof. Suppose z_0 is a boundary point of set S. Then, by definition, z_0 is neither an interior point of S nor an exterior point of S. So z_0 has no neighborhoods which are subsets of S and z_0 has no neighborhoods which are disjoint from S. So each neighborhood of z_0 must contain a point of S and also contain a point not in S, as claimed.

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Now suppose every neighborhood of z_0 contains at least one point in set S and at least one point not in S. Then no neighborhood of z_0 is a subset of set S and hence z_0 is not an interior point of S. Also, no neighborhood of z_0 is disjoint with set S and hence z_0 is not an exterior point of S. So, by definition, z_0 is a boundary point of S. □

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Lemma 1.11.B

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Proof. Let z_0 be an accumulation point of set S. ASSUME $z_0 \notin S$. Then each deleted neighborhood of z_0 contains at least one point of S, and so each neighborhood of z_0 contains both a point in S and a point not in S (namely z_0 itself).

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