Complex Variables

Chapter 1. Complex Numbers

Section 1.2. Basic Algebraic Properties—Proofs of Theorems

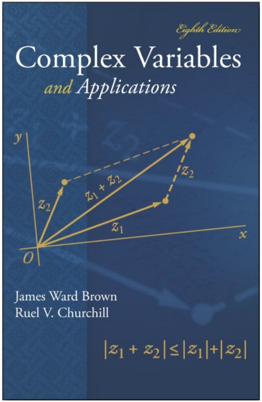

Table of contents

Theorem 1.2.1

Theorem 1.2.1. For any $z_1, z_2, z_3 \in \mathbb{C}$ we have the following.

1. Commutivity of addition and multiplication:

$$z_1 + z_2 = z_2 + z_1 \text{ and } z_1 z_2 = z_2 z_1.$$

2. Associativity of addition and multiplication:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ and } (z_1 z_2) z_3 = z_1 (z_2 z_3).$$

3. Distribution of multiplication over addition:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

4. There is an *additive identity* $0 = 0 + i0$ such that $0 + z = z$ for all $z \in \mathbb{C}$. There is a *multiplicative identity* $1 = 1 + i0$ such that $z1 = z$ for all $z \in \mathbb{C}$. Also, $z0 = 0$ for all $z \in \mathbb{C}$.
5. For each $z \in \mathbb{C}$ there is $z' \in \mathbb{C}$ such that $z' + z = 0$. z' is the *additive inverse* of z (denoted $-z$). If $z \neq 0$, then there is $z'' \in \mathbb{C}$ such that $z''z = 1$. z'' is the *multiplicative inverse* of z (denoted z^{-1}).

Theorem 1.2.1 (continued 1)

Proof. We have abandoned the ordered pair notation, so let $z_k = x_k + iy_k$ for $k = 1, 2, 3$ and let $z = x + iy$, where $x_k, y_k, x, y \in \mathbb{R}$.

1. (Commutivity) For addition, we have

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \text{ by the definition of } + \text{ in } \mathbb{C} \\ &= (x_2 + x_1) + i(y_2 + y_1) \text{ since } + \text{ is commutative in } \mathbb{R} \\ &= (x_2 + iy_2) + (x_1 + iy_1) \text{ by the definition of } + \text{ in } \mathbb{C} \\ &= z_2 + z_1. \end{aligned}$$

Theorem 1.2.1 (continued 2)

Proof (continued). For multiplication we have

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= (x_2x_1 - y_2y_1) + i(y_2x_1 + x_2y_1) \\ &\quad \text{since } \cdot \text{ and } + \text{ are commutative in } \mathbb{R} \\ &= (x_2 + iy_2)(x_1 + iy_1) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= z_2 z_1. \end{aligned}$$

2. (Associativity) For addition we have

$$\begin{aligned} (z_1 + z_2) + z_3 &= ((x_1 + iy_1) + (x_2 + iy_2)) + (x_3 + iy_3) \\ &= ((x_1 + x_2) + i(y_1 + y_2)) + (x_3 + iy_3) \\ &\quad \text{by the definition of } + \text{ in } \mathbb{C} \end{aligned}$$

Theorem 1.2.1 (continued 2)

Proof (continued). For multiplication we have

$$\begin{aligned} z_1z_2 &= (x_1+iy_1)(x_2+iy_2) \\ &= (x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= (x_2x_1-y_2y_1)+i(y_2x_1+x_2y_1) \\ &\quad \text{since } \cdot \text{ and } + \text{ are commutative in } \mathbb{R} \\ &= (x_2+iy_2)(x_1+iy_1) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= z_2z_1. \end{aligned}$$

2. (Associativity) For addition we have

$$\begin{aligned} (z_1+z_2)+z_3 &= ((x_1+iy_1)+(x_2+iy_2))+(x_3+iy_3) \\ &= ((x_1+x_2)+i(y_1+y_2))+(x_3+iy_3) \\ &\quad \text{by the definition of } + \text{ in } \mathbb{C} \end{aligned}$$

Theorem 1.2.1 (continued 3)

$$
\begin{aligned}
(z_1 + z_2) + z_3 &= ((x_1 + x_2) + x_3) + i((y_1 + y_2) + y_3) \\
&\quad \text{by the definition of } + \text{ in } \mathbb{C} \\
&= (x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3)) \\
&\quad \text{since } + \text{ is associative in } \mathbb{R} \\
&= (x_1 + iy_1) + ((x_2 + x_3) + i(y_2 + y_3)) \\
&\quad \text{by the definition of } + \text{ in } \mathbb{C} \\
&= z_1 + (z_2 + z_3)
\end{aligned}
$$

For multiplication we have

$$
\begin{aligned}
(z_1 z_2) z_3 &= ((x_1 + iy_1)(x_2 + iy_2))(x_3 + iy_3) \\
&= ((x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2))(x_3 + iy_3) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C}
\end{aligned}
$$

Theorem 1.2.1 (continued 3)

$$\begin{aligned}
(z_1 + z_2) + z_3 &= ((x_1 + x_2) + x_3) + i((y_1 + y_2) + y_3) \\
&\quad \text{by the definition of } + \text{ in } \mathbb{C} \\
&= (x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3)) \\
&\quad \text{since } + \text{ is associative in } \mathbb{R} \\
&= (x_1 + iy_1) + ((x_2 + x_3) + i(y_2 + y_3)) \\
&\quad \text{by the definition of } + \text{ in } \mathbb{C} \\
&= z_1 + (z_2 + z_3)
\end{aligned}$$

For multiplication we have

$$\begin{aligned}
(z_1 z_2) z_3 &= ((x_1 + iy_1)(x_2 + iy_2))(x_3 + iy_3) \\
&= ((x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2))(x_3 + iy_3) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C}
\end{aligned}$$

Theorem 1.2.1 (continued 4)

$$
\begin{aligned}
(z_1 z_2) z_3 &= ((x_1 x_2 - y_1 y_2) x_3 - (y_1 x_2 + x_1 y_2) y_3) + \\
&\quad i((y_1 x_2 + x_1 y_2) x_3 + (x_1 x_2 - y_1 y_2) y_3) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C} \\
&= (x_1(x_2 x_3 - y_2 y_3) - y_1(y_2 x_3 + x_2 y_3)) \\
&\quad + i(y_1(x_2 x_3 - y_2 y_3) + x_1(y_2 x_3 + x_2 y_3)) \\
&\quad \text{by distribution, commutivity, and associativity in } \mathbb{R} \\
&= (x_1 + i y_1)((x_2 x_3 - y_2 y_3) + i(y_2 x_3 + x_2 y_3)) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C} \\
&= (x_2 + i y_1)((x_2 + i y_2)(x_3 + i y_3)) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C} \\
&= z_1(z_2 z_3).
\end{aligned}
$$

Theorem 1.2.1 (continued 5)

3. (Distribution) For distribution we have

$$
\begin{aligned}
z_1(z_2+z_3) &= (z_1+iy_1)((x_2+iy_2)+(x_3+iy_3)) \\
&= (x_1+iy_1)((x_2+x_3)+i(y_2+y_3)) \\
&\quad \text{by the definition of } + \text{ in } \mathbb{C} \\
&= (x_1(x_2+x_3)-y_1(y_2+y_3))+i(y_1(x_2+x_3)+x_1(y_2+y_3)) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C} \\
&= (x_1x_2+x_1x_3-y_1y_2-y_1y_3)+i(y_1x_2+y_1x_3+x_1y_2+x_1y_3) \\
&\quad \text{by distribution in } \mathbb{R} \\
&= ((x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2))+((x_1x_3-y_1y_3) \\
&\quad +i(y_1x_3+x_1y_3) \text{ by the definition of } + \text{ in } \mathbb{C} \\
&= (x_1+iy_1)(x_2+iy_2)+(x_1+iy_1)(x_3+iy_3) \\
&\quad \text{by the definition of } \cdot \text{ in } \mathbb{C} \\
&= z_1z_2+z_1z_3.
\end{aligned}
$$

Theorem 1.2.1 (continued 6)

4. (Identities) We easily have

$$\begin{aligned} 0+z &= (0+i0)+(x+iy) \\ &= (0+x)+i(0+y) \text{ by the definition of } + \text{ in } \mathbb{C} \\ &= x+iy \text{ since 0 is the additive identity in } \mathbb{R} \\ &= z \end{aligned}$$

and

$$\begin{aligned} 1z &= (1+i0)(x+iy) \\ &= ((1)(x)-(0)(y))+i((0)(x)+(1)(y)) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= x+iy \text{ since 1 is the multiplicative identity in } \mathbb{R} \\ &= z. \end{aligned}$$

Theorem 1.2.1 (continued 6)

4. (Identities) We easily have

$$\begin{aligned} 0 + z &= (0 + i0) + (x + iy) \\ &= (0 + x) + i(0 + y) \text{ by the definition of } + \text{ in } \mathbb{C} \\ &= x + iy \text{ since 0 is the additive identity in } \mathbb{R} \\ &= z \end{aligned}$$

and

$$\begin{aligned} 1z &= (1 + i0)(x + iy) \\ &= ((1)(x) - (0)(y)) + i((0)(x) + (1)(y)) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= x + iy \text{ since 1 is the multiplicative identity in } \mathbb{R} \\ &= z. \end{aligned}$$

Theorem 1.2.1 (continued 6)

Also,

$$\begin{aligned} z0 &= (x+iy)(0+i0) \\ &= ((x)(0)-(y)(0)) + i((y)(0)+(x)(0)) \text{ by the definition of } \cdot \text{ in } \mathbb{C} \\ &= 0+i0 \text{ since } r0 = 0 \text{ for all } r \in \mathbb{R} \\ &= 0. \end{aligned}$$

5. (Inverses) For $z = x + iy$, we take $z' = (-x) + i(-y)$ and then

$$\begin{aligned} z + z' &= (x+iy) + ((-x) + i(-y)) \\ &= (x + (-x)) + i(y + (-y)) \text{ by the definition of } + \text{ in } \mathbb{C} \\ &= 0 + i0 \text{ since } -x \text{ and } -y \text{ are the } + \text{ inverses of } x \text{ and } y, \\ &\quad \text{respectively, in } \mathbb{R} \\ &= 0. \end{aligned}$$

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Also,

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5. (Inverses) For $z = x + iy$, we take $z' = (-x) + i(-y)$ and then

$$\begin{aligned} z+z' &= (x+iy)+((-x)+i(-y)) \\ &= (x+(-x))+i(y+(-y)) \text{ by the definition of } + \text{ in } \mathbb{C} \\ &= 0+i0 \text{ since } -x \text{ and } -y \text{ are the } + \text{ inverses of } x \text{ and } y, \\ &\quad\ \text{respectively, in } \mathbb{R} \\ &= 0. \end{aligned}$$

Theorem 1.2.1 (continued 6)

For $z = x + iy \neq 0$, take $z'' = x/(x^2 + y^2) + i(-y)/(x^2 + y^2)$ (see page 4 of the text for motivation). Then

$$\begin{aligned} zz'' &= (x + iy)(x/(x^2 + y^2) + i(-y)/(x^2 + y^2)) \\ &= ((x)(x/(x^2 + y^2) - (y)((-y)/x^2 + y^2))) \\ &\quad +i((y)(x/(x^2 + y^2)) + x((-y)/(x^2 + y^2))) \\ &\quad \text{by the definition of } \cdot \text{ in } \mathbb{C} \\ &= 1 + i0 \text{ by the multiplicative and additive inverse properties in } \mathbb{R} \\ &= 1. \end{aligned}$$

□

Corollary 1.2.2

Corollary 1.2.2. For $z_1, z_2, z_3 \in \mathbb{C}$ if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$. That is, $\mathbb{C}$ has no "zero divisors."

Proof. Suppose one of z_1 or z_2 is nonzero. WLOG, say $z_1 \neq 0$. Then, by Theorem 1.2.1(5), there is a multiplicative inverse $z_1^{-1} \in \mathbb{C}$ such that $z_1 z_1^{-1} = 1$.

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$$\begin{aligned} z_2 &= z_2 1 \text{ by Theorem 1.2.1(4) } (\cdot \text{ identity}) \\ &= z_2(z_1 z_1^{-1}) \\ &= (z_1 z_1^{-1}) z_2 = (z_1^{-1} z_1) z_2 \text{ by Theorem 1.2.1(1) (commutivity of } \cdot) \\ &= z_1^{-1}(z_1 z_2) \text{ by Theorem 1.2.1(2) (associativity of } \cdot) \\ &= z_1^{-1}(0) \text{ by hypothesis} \\ &= 0 \text{ by Theorem 1.2.1(4).} \end{aligned}$$

So if one of z_1, z_2 is nonzero, then the other is 0 and the result follows. □

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$$\begin{aligned} z_2 &= z_2 1 \text{ by Theorem 1.2.1(4) } (\cdot \text{ identity}) \\ &= z_2(z_1 z_1^{-1}) \\ &= (z_1 z_1^{-1}) z_2 = (z_1^{-1} z_1) z_2 \text{ by Theorem 1.2.1(1) (commutivity of } \cdot) \\ &= z_1^{-1}(z_1 z_2) \text{ by Theorem 1.2.1(2) (associativity of } \cdot) \\ &= z_1^{-1}(0) \text{ by hypothesis} \\ &= 0 \text{ by Theorem 1.2.1(4).} \end{aligned}$$

So if one of z_1, z_2 is nonzero, then the other is 0 and the result follows. □