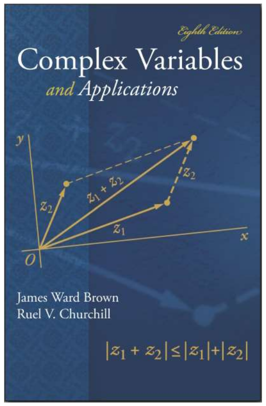
Complex Variables

Chapter 1. Complex Numbers

Section 1.3. Further Properties—Proofs of Theorems



Lemma 1.3.1

Lemma 1.3.1. For any $z_1, z_2, z_3 \in \mathbb{C}$, with $z_3 \neq 0$, we have

$$\frac{z_1}{z_3} + \frac{z_2}{z_3} = \frac{z_1 + z_2}{z_3}.$$

Proof. We have

$$\begin{aligned} \frac{z_1 + z_2}{z_3} &= (z_1 + z_2) z_3^{-1} \text{ by the definition of division in } \mathbb{C} \\ &= z_1 z_3^{-1} + z_2 z_3^{-1} \text{ by Theorem 1.2.1(3) (distribution)} \\ &= \frac{z_1}{z_3} + \frac{z_2}{z_3} \text{ by the definition of division in } \mathbb{C}. \end{aligned}$$

□

Lemma 1.3.2

Lemma 1.3.2. If $z_1, z_2 \in \mathbb{C}$, with $z_1 \neq 0$, $z_2 \neq 0$, then

$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = \frac{1}{z_1 z_2},$$

or equivalently, $z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1}$.

Proof. We have

$$\begin{aligned} (z_1 z_2)(z_1^{-1} z_2^{-1}) &= (z_1 z_2)(z_2^{-1} z_1^{-1}) \text{ by Theorem 1.2.1(1), commutativity} \\ &= ((z_1 z_2) z_2^{-1}) z_1^{-1} \text{ by Theorem 1.2.1(2), associativity} \\ &= (z_1 (z_2 z_2^{-1})) z_1^{-1} \text{ by Theorem 1.2.1(2), associativity} \\ &= (z_1(1)) z_1^{-1} \text{ by the definition of multiplicative inverse} \\ &= z_1 z_1^{-1} \text{ since 1 is the multiplicative identity} \\ &= 1 \text{ by the definition of multiplicative inverse.} \end{aligned}$$

Since $(z_1 z_2)(z_1^{-1} z_2^{-1}) = 1$ then $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$. □