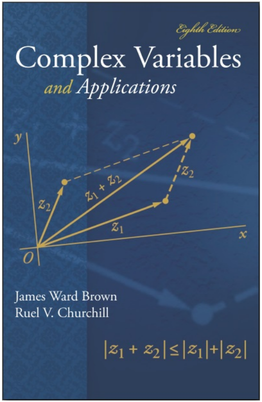
Complex Variables

Chapter 1. Complex Numbers

Section 1.3. Further Properties—Proofs of Theorems



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Lemma 1.3.1

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$$\frac{z_1}{z_3} + \frac{z_2}{z_3} = \frac{z_1 + z_2}{z_3}.$$

Proof. We have

$$\begin{aligned}
\frac{z_1 + z_2}{z_3} &= (z_1 + z_2)z_3^{-1} \text{ by the definition of division in } \mathbb{C} \\
&= z_1 z_3^{-1} + z_2 z_3^{-1} \text{ by Theorem 1.2.1(3) (distribution)} \\
&= \frac{z_1}{z_3} + \frac{z_2}{z_3} \text{ by the definition of division in } \mathbb{C}.
\end{aligned}$$

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Lemma 1.3.2

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$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = \frac{1}{z_1 z_2},$$

or equivalently, $z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1}$.

Proof. We have

$$\begin{aligned}
(z_1z_2)(z_1^{-1}z_2^{-1}) &= (z_1z_2)(z_2^{-1}z_1^{-1}) \text{ by Theorem 1.2.1(1), commutativity} \\
&= ((z_1z_2)z_2^{-1})z_1^{-1} \text{ by Theorem 1.2.1(2), associativity} \\
&= (z_1(z_2z_2^{-1}))z_1^{-1} \text{ by Theorem 1.2.1(2), associativity} \\
&= (z_1(1))z_1^{-1} \text{ by the definition of multiplicative inverse} \\
&= z_1z_1^{-1} \text{ since 1 is the multiplicative identity} \\
&= 1 \text{ by the definition of multiplicative inverse.}
\end{aligned}$$

Since $(z_1z_2)(z_1^{-1}z_2^{-1}) = 1$ then $(z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$. □

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