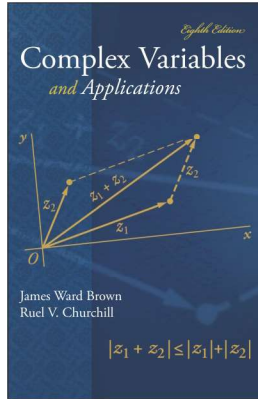


# Complex Variables

## Chapter 1. Complex Numbers

### Section 1.5. Complex Conjugates—Proofs of Theorems



## Theorem 1.5.1

**Theorem 1.5.1.** For all  $z_1, z_2 \in \mathbb{C}$  we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} & \overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1} \overline{z_2} & \overline{z_1/z_2} &= \overline{z_1}/\overline{z_2} \\ \operatorname{Re}(z) &= (z + \overline{z})/2 & \operatorname{Im}(z) &= (z - \overline{z})/(2i) \end{aligned}$$

**Proof.** Let  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ . Then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) = (x_1 + x_2) + i(-y_1 - y_2) \\ &= (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}. \end{aligned}$$

Similarly

$$\begin{aligned} \overline{z_1 - z_2} &= \overline{(x_1 + iy_1) - (x_2 + iy_2)} = \overline{(x_1 - x_2) + i(y_1 - y_2)} \\ &= (x_1 - x_2) - i(y_1 - y_2) = (x_1 - x_2) + i(-y_1 + y_2) \\ &= (x_1 - iy_1) - (x_2 - iy_2) = \overline{z_1} - \overline{z_2}. \end{aligned}$$

## Theorem 1.5.1 (continued 1)

**Proof (continued).** Next,

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)} \\ &= (x_1 x_2 - y_1 y_2) - i(y_1 x_2 + x_1 y_2) \\ &= (x_1 x_2 - (-y_1)(-y_2)) + i((-y_1)(x_2) + (x_1)(-y_2)) = (x_1 + i(-y_1))(x_2 + i(-y_2)) \\ &= (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1} \overline{z_2}. \end{aligned}$$

Also,

$$\begin{aligned} \overline{z_1/z_2} &= \overline{(x_1 + iy_1)/(x_2 + iy_2)} = \frac{\overline{x_1 x_2 + y_1 y_2}}{x_2^2 + y_2^2} + i \frac{\overline{y_1 x_2 - x_1 y_2}}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} - i \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + (-y_1)(-y_2)}{x_2^2 + (-y_2)^2} + i \frac{(-y_1)(x_2) - (x_1)(-y_2)}{x_2^2 + (-y_2)^2} = \overline{z_1}/\overline{z_2}. \end{aligned}$$

## Theorem 1.5.1 (continued 2)

**Proof (continued).** Next, let  $z = x + iy = \operatorname{Re}(z) + i\operatorname{Im}(z)$ . Then

$$\begin{aligned} (z + \overline{z})/2 &= ((x + iy) + \overline{(x + iy)})/2 \\ &= ((x + iy) + (x - iy))/2 = (2x + i0)/2 = x = \operatorname{Re}(z). \end{aligned}$$

Finally,

$$\begin{aligned} (z - \overline{z})/(2i) &= ((x + iy) - \overline{(x + iy)})/(2i) \\ &= ((x + iy) - (x - iy))/(2i) = (0 + i(2y))/(2i) = y = \operatorname{Im}(z). \end{aligned}$$

□

## Theorem 1.5.2

**Theorem 1.5.2.** For all  $z_1, z_2 \in \mathbb{C}$  we have

$$|z_1 z_2| = |z_1| |z_2| \text{ and } \left| \frac{z_1}{z_2} \right| = |z_1| / |z_2| \text{ (for } z \neq 0 \text{)}.$$

**Proof.** We have

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2) \overline{(z_1 z_2)} \text{ by Note 1.5.A} \\ &= z_1 z_2 \bar{z}_1 \bar{z}_2 \text{ by Theorem 1.5.1} \\ &= z_1 \bar{z}_1 z_2 \bar{z}_2 \text{ by Commutativity of Multiplication, Theorem 1.2.1(1)} \\ &= |z_1|^2 |z_2|^2 \text{ by Note 1.5.A.} \end{aligned}$$

Taking square roots (and since moduli are nonnegative),  $|z_1 z_2| = |z_1| |z_2|$ .  
The second equation is addressed in Exercise 1.5.5 (in Exercise 1.6.5 in the 9th edition of the book).  $\square$