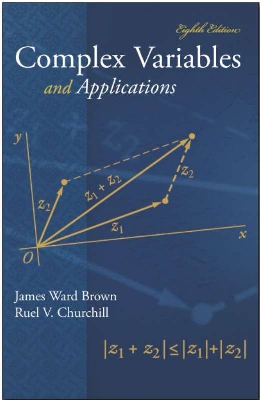
Complex Variables

Chapter 1. Complex Numbers

Section 1.5. Complex Conjugates—Proofs of Theorems



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Theorem 1.5.1

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$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 \qquad \overline{z_1 / z_2} = \overline{z}_1 / \overline{z}_2$$
$$\mathrm{Re}(z) = (z + \overline{z})/2 \qquad \mathrm{Im}(z) = (z - \overline{z})/(2i)$$

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$$
$$= (x_1 + x_2) - i(y_1 + y_2) = (x_1 + x_2) + i(-y_1 + -y_2)$$
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Theorem 1.5.1 (continued 1)

Proof (continued). Next,

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)}$$

$$= (x_1x_2 - y_1y_2) - i(y_1x_2 + x_1y_2)$$

$$= (x_1x_2 - (-y_1)(-y_2)) + i((-y_1)(x_2) + (x_1)(-y_2)) = (x_1 + i(-y_1))(x_2 + i(-y_2))$$

$$= (x_1 - iy_1)(x_2 - iy_2) = \overline{z}_1 \overline{z}_2.$$

Also,

$$\overline{z_1/z_2} = \overline{(x_1 + iy_1)/(x_2 + iy_2)} = \overline{\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

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Proof (continued). Next, let $z = x + iy = \text{Re}(z) + i\text{Im}(z)$. Then

$$(z + \overline{z})/2 = ((x + iy) + \overline{(x + iy)})/2$$

$$= ((x + iy) + (x - iy))/2 = (2x + i0)/2 = x = \text{Re}(z).$$

Finally,

$$(z - \overline{z})/(2i) = ((x + iy) - \overline{(x + iy)})/(2i)$$

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$$|z_1 z_2| = |z_1||z_2| \text{ and } \left|\frac{z_1}{z_2}\right| = |z_1|/|z_2| \text{ (for } z \neq 0).$$

Proof. We have

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)\overline{(z_1 z_2)} \text{ by Note 1.5.A} \\ &= z_1 z_2 \bar{z}_1 \bar{z}_2 \text{ by Theorem 1.5.1} \\ &= z_1 \bar{z}_1 z_2 \bar{z}_2 \text{ by Commutativity of Multiplication, Theorem 1.2.1(1)} \\ &= |z_1|^2 |z_2|^2 \text{ by Note 1.5.A.} \end{aligned}$$

Taking square roots (and since moduli are nonnegative), $|z_1 z_2| = |z_1||z_2|$. The second equation is addressed in Exercise 1.5.5 (in Exercise 1.6.5 in the 9th edition of the book). □

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