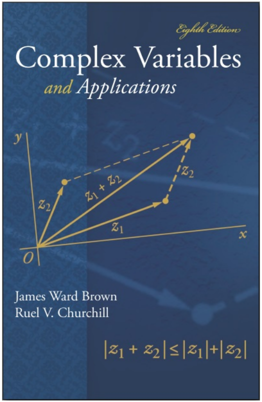
Complex Variables

Chapter 1. Complex Numbers

Section 1.7. Products and Powers in Exponential Form—Proofs of Theorems



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Theorem 1.7.1

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$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \text{ and } \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Proof. First, notice that

$$\begin{aligned}
e^{i\theta_1} e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\
&= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\
&\quad \text{since } \cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \\
&\quad \text{and } \sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \\
&= e^{i(\theta_1 + \theta_2)}.
\end{aligned}$$

So

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i\theta_1} e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)}.$$

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Proof. First, notice that

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &\quad \text{since } \cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \\ &\quad \text{and } \sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \\ &= e^{i(\theta_1+\theta_2)}. \end{aligned}$$

So

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Theorem 1.7.1 (continued 1)

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Proof (continued). Next,

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \frac{e^{i\theta_1} e^{-i\theta_2}}{e^{i\theta_2} e^{-i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

□

Corollary 1.7.2

Corollary 1.7.2. If $z = re^{i\theta} \in \mathbb{C}$, then for $n \in \mathbb{Z}$ we have $z^n = r^n e^{in\theta}$.

Proof. For $n > 0$, we use mathematical induction. First, for the base case $n = 1$, the result is trivial. Now suppose the result holds for $n = m$; that is, suppose $z^m = r^m e^{im\theta}$ (this is the "induction hypothesis"). To complete the induction argument, we must show the result holds for $n = m + 1$.

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$$\begin{aligned} z^{m+1} &= z^m z = (r^m e^{im\theta}) z \text{ by the induction hypothesis} \\ &= r^m e^{im\theta} r e^{i\theta} = r^m r e^{im\theta} e^{im\theta} \\ &= r^{m+1} e^{i(m+1)\theta} \text{ by Theorem 1.7.1.} \end{aligned}$$

So the result holds for all $n > 0$.

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So the result holds for all $n > 0$.

For $n = 0$, we have $z^0 = 1$ (by "convention," provided $z \neq 0$) and $1 = r^0 e^{i0}$, so the result holds for $n = 0$.

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Corollary 1.7.2 (continued)

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Proof (continued). For $n < 0$, let $m = -n$ (so $m > 0$) and

$$\begin{aligned} z^n &= z^{-m} = (z^{-1})^m = (r^{-1}e^{-i\theta})^m \text{ by Note 1.7.A} \\ &= r^{-m}(e^{-i\theta})^m = \left(\frac{1}{r}\right)^m \frac{1}{(e^{i\theta})^m} \\ &= \frac{1}{r^m}\frac{1}{e^{im\theta}} \text{ by the first part of the proof, since } m > 0 \\ &= r^{-m}e^{i(-m)\theta} = r^n e^{in\theta}. \end{aligned}$$

So the result holds for all $n < 0$ and hence holds for all $n \in \mathbb{Z}$. □

Corollary 1.7.3

Corollary 1.7.3. For all $n \in \mathbb{Z}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. Since $e^{i\theta} = \cos\theta + i\sin\theta$, then

$$\begin{aligned} (e^{i\theta})^n &= (\cos\theta + i\sin\theta)^n \\ &= e^{in\theta} \text{ by Corollary 1.7.2} \\ &= \cos(n\theta) + i\sin(n\theta). \end{aligned}$$

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$\square$