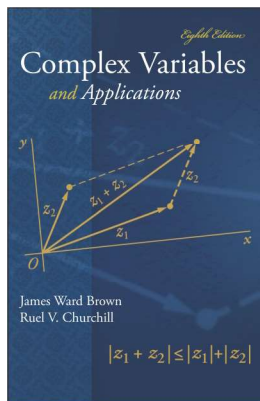


# Complex Variables

## Chapter 1. Complex Numbers

### Section 1.8. Arguments of Products and Quotients—Proofs of Theorems



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Lemma 1.8.1

## Lemma 1.8.1

**Lemma 1.8.1.** For any  $z_1, z_2 \in \mathbb{C}$  with  $z_1 \neq 0, z_2 \neq 0$ , we have  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ .

**Proof.** Let  $\theta_1 + \theta_2 \in \arg(z_1) + \arg(z_2)$  (so that  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ ). Then  $z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$  by Theorem 1.7.1, and so  $\theta_1 + \theta_2 \in \arg(z_1 z_2)$ . So  $\arg(z_1) + \arg(z_2) \subset \arg(z_1 z_2)$ .

Now let  $\theta \in \arg(z_1 z_2)$ . Then  $z_1 z_2 = r_1 r_2 e^{i\theta}$  where  $r_1 = |z_1|$  and  $r_2 = |z_2|$ . Let  $\theta_1 \in \arg(z_1)$ . Then  $z_1 = r_1 e^{i\theta_1}$ . Let  $\theta_2 = \theta - \theta_1$  so that, by Theorem 1.7.1,

$$r_2 e^{i\theta_2} = r_2 e^{i(\theta - \theta_1)} = r_2 e^{i\theta} e^{-i\theta_1} = \frac{r_1 r_2 e^{i\theta}}{r_1 e^{i\theta_1}} = \frac{z_1 z_2}{z_1} = z_2$$

and so  $\theta_2 \in \arg(z_2)$ . Therefore  $\theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2)$ . Similarly, if  $\theta_2 \in \arg(z_2)$  then  $\theta_1 = \theta - \theta_2 \in \arg(z_1)$  and  $\theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2)$ . That is,  $\arg(z_1 z_2) \subset \arg(z_1) + \arg(z_2)$ . Hence,  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ . □

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Lemma 1.8.2

## Lemma 1.8.2

**Lemma 1.8.2.** For any  $z_1, z_2 \in \mathbb{C}$  with  $z_1 \neq 0, z_2 \neq 0$  we have  $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$ .

**Proof.** First, we have for  $z_2 = r_2 e^{i\theta_2}$  that  $z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2}$  (by Note 1.7.A), so  $\arg(z_2^{-1}) = -\arg(z_2)$ . Now by Lemma 1.8.1,

$$\arg(z_1/z_2) = \arg(z_1 z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) - \arg(z_2).$$

□

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