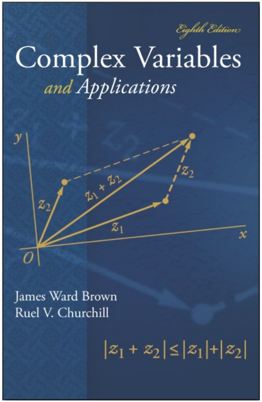
Complex Variables

Chapter 1. Complex Numbers

Section 1.8. Arguments of Products and Quotients—Proofs of Theorems



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Lemma 1.8.1

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Proof. Let $\theta_1 + \theta_2 \in \arg(z_1) + \arg(z_2)$ (so that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$). Then $z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$ by Theorem 1.7.1, and so $\theta_1 + \theta_2 \in \arg(z_1 z_2)$. So $\arg(z_1) + \arg(z_2) \subset \arg(z_1 z_2)$.

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Now let $\theta \in \arg(z_1 z_2)$. Then $z_1 z_2 = r_1 r_2 e^{i\theta}$ where $r_1 = |z_1|$ and $r_2 = |z_2|$. Let $\theta_1 \in \arg(z_1)$. Then $z_1 = r_1 e^{i\theta_1}$. Let $\theta_2 = \theta - \theta_1$ so that, by Theorem 1.7.1,

$$r_2 e^{i\theta_2} = r_2 e^{i(\theta - \theta_1)} = r_2 e^{i\theta} e^{-i\theta_1} = \frac{r_1 r_2 e^{i\theta}}{r_1 e^{i\theta_1}} = \frac{z_1 z_2}{z_1} = z_2$$

and so $\theta_2 \in \arg(z_2)$.

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and so $\theta_2 \in \arg(z_2)$. Therefore $\theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2)$. Similarly, if $\theta_2 \in \arg(z_2)$ then $\theta_1 = \theta - \theta_2 \in \arg(z_1)$ and $\theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2)$. That is, $\arg(z_1 z_2) \subset \arg(z_1) + \arg(z_2)$. Hence, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. □

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Lemma 1.8.2

Lemma 1.8.2. For any $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq 0$, $z_2 \neq 0$ we have $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.

Proof. First, we have for $z_2 = r_2 e^{i\theta_2}$ that $z_2^{-1} = \dfrac{1}{r_2} e^{-i\theta_2}$ (by Note 1.7.A), so $\arg(z_2^{-1}) = -\arg(z_2)$. Now by Lemma 1.8.1,

$$\arg(z_1/z_2) = \arg(z_1 z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) - \arg(z_2).$$

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