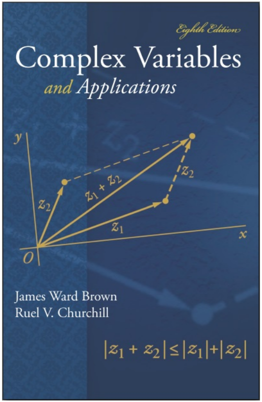
Complex Variables

Chapter 1. Complex Numbers

Section 2.15. Limits—Proofs of Theorems



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Proof. Let $\varepsilon > 0$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $0 < |z - z_0| < \delta_1$ implies $|f(z) - w_0| < \varepsilon/2$, and $0 < |z - z_0| < \delta_2$ implies $|f(z) - w_1| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $0 < |z - z_0| < \delta$ then

$$\begin{aligned} |w_1 - w_0| &= |(f(z) - w_0) - (f(z) - w_1)| \\ &\leq |f(z) - w_0| + |f(z) - w_1| \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, then it must be that $w_0 = w_1$, as claimed. □

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