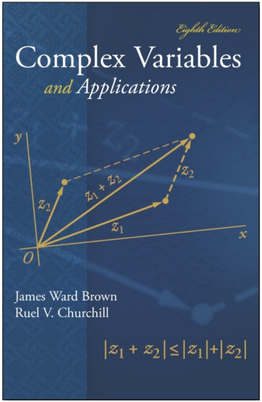
Complex Variables

Chapter 2. Analytic Functions

Section 2.16. Theorems on Limits—Proofs of Theorems



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Theorem 2.16.1

Theorem 2.16.1. Suppose that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

Proof. First, suppose $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. Let $\varepsilon > 0$.

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Proof. First, suppose $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. Let $\varepsilon > 0$. Then there is $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \text{ implies } |u - u_0| < \varepsilon/2$$

and

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2 \text{ implies } |v - v_0| < \varepsilon/2.$$

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Theorem 2.16.1 (continued 1)

Proof (continued). Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |z - z_0| < \delta$ implies

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)|$$

$$= |(x + iy) - (x_0 + iy_0)| = |z - z_0| < \delta$$

and, since $\delta \le \delta_1$ and $\delta \le \delta_2$, we have both $|u - u_0| < \varepsilon/2$ and $|v - v_0| < \varepsilon/2$, or

Theorem 2.16.1 (continued 1)

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$$= |(x+iy) - (x_0 + iy_0)| = |z - z_0| < \delta$$

and, since $\delta \le \delta_1$ and $\delta \le \delta_2$, we have both $|u - u_0| < \varepsilon/2$ and $|v - v_0| < \varepsilon/2$, or

$$\begin{aligned} |f(z) - w_0| &= |(u+iv) - (u_0 + iv_0)| = |(u-u_0) + i(v-v_0)| \\ &\le |u-u_0| + |v-v_0| \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So $\lim_{z \to z_0} f(z) = w_0$.

Theorem 2.16.1 (continued 1)

Proof (continued). Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |z - z_0| < \delta$ implies

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)|$$

$$= |(x + iy) - (x_0 + iy_0)| = |z - z_0| < \delta$$

and, since $\delta \leq \delta_1$ and $\delta \leq \delta_2$, we have both $|u - u_0| < \varepsilon/2$ and $|v - v_0| < \varepsilon/2$, or

$$\begin{aligned} |f(z) - w_0| &= |(u + iv) - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| \\ &\leq |u - u_0| + |v - v_0| \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So $\lim_{z \to z_0} f(z) = w_0$.

Theorem 2.16.1 (continued 2)

Proof (continued). Second, suppose $\lim_{z\to z_0} f(z) = w_0$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $0 < |z - z_0| = |(x + iy) - (x_0 + iy_0)| < \delta$ implies $|f(z) - w_0| = |(u + iv) - (u_0 + iv_0)| < \varepsilon$. This implies,

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)|$$

$$= |(x + iy) - (x_0 + iy_0)| = |z - z_0| < \delta$$

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$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} = |(x-x_0) + i(y-y_0)|$$

$$= |(x+iy) - (x_0+iy_0)| = |z - z_0| < \delta$$

and so

$$|u - u_0| \le \sqrt{(u-u_0)^2 + (v-v_0)^2}$$

$$= |(u-u_0) + i(v-v_0)| = |(u+iv) - (u_0+iv_0)| < \varepsilon$$

and

$$|v - v_0| \le \sqrt{(u-u_0)^2 + (v-v_0)^2}$$

$$= |(u-u_0) + i(v-v_0)| = |(u+iv) - (u_0+iv_0)| < \varepsilon.$$

That is, $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = v_0$. □

Theorem 2.16.1 (continued 2)

Proof (continued). Second, suppose $\lim_{z \to z_0} f(z) = w_0$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $0 < |z - z_0| = |(x + iy) - (x_0 + iy_0)| < \delta$ implies $|f(z) - w_0| = |(u + iv) - (u_0 + iv_0)| < \varepsilon$. This implies,

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)|$$

$$= |(x + iy) - (x_0 + iy_0)| = |z - z_0| < \delta$$

and so

$$|u - u_0| \leq \sqrt{(u - u_0)^2 + (v - v_0)^2}$$

$$= |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| < \varepsilon$$

and

$$|v - v_0| \leq \sqrt{(u - u_0)^2 + (v - v_0)^2}$$

$$= |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| < \varepsilon.$$

That is, $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = v_0$. □

Theorem 2.16.2

Theorem 2.16.2. Suppose that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$. Then

$$\begin{aligned} \lim_{z\to z_0} (f(z) + F(z)) &= w_0 + W_0 \\ \lim_{z\to z_0} f(z)F(z) &= w_0 W_0, \text{ and} \\ \lim_{z\to z_0} f(z)/F(z) &= w_0/W_0 \text{ if } W_0 \neq 0 \end{aligned}$$

Proof. Let $f(z) = u(x,y) + iv(x,y)$ and $F(z) = U(x,y) + iV(x,y)$ where $z = x_i y$, $z_0 = x_0 + iv_0$, $w_0 = u_0 + iv_0$, and $W_0 = U_0 + iV_0$.

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$$\begin{aligned}\lim_{z\to z_0}(f(z)+F(z)) &= w_0 + W_0\\ \lim_{z\to z_0} f(z)F(z) &= w_0 W_0, \text{ and}\\ \lim_{z\to z_0} f(z)/F(z) &= w_0/W_0 \text{ if } W_0 \neq 0\end{aligned}$$

Proof. Let $f(z) = u(x,y) + iv(x,y)$ and $F(z) = U(x,y) + iV(x,y)$ where $z = x_i y$, $z_0 = x_0 + iv_0$, $w_0 = u_0 + iv_0$, and $W_0 = U_0 + iV_0$. Since by hypotheses, $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$, then by Theorem 2.16.1 we have $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$, $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$, $\lim_{(x,y)\to(x_0,y_0)} U(x,y) = U_0$, and $\lim_{(x,y)\to(x_0,y_0)} V(x,y) = V_0$.

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$$\begin{aligned} \lim_{z\longrightarrow z_0}(f(z)+F(z)) &= w_0+W_0 \\ \lim_{z\longrightarrow z_0} f(z)F(z) &= w_0W_0, \text{ and} \\ \lim_{z\longrightarrow z_0} f(z)/F(z) &= w_0/W_0 \text{ if } W_0\neq 0 \end{aligned}$$

Proof. Let $f(z) = u(x,y) + iv(x,y)$ and $F(z) = U(x,y) + iV(x,y)$ where $z = x_iy$, $z_0 = x_0 + iv_0$, $w_0 = u_0 + iv_0$, and $W_0 = U_0 + iV_0$. Since by hypotheses, $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$, then by Theorem 2.16.1 we have $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$, $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$, $\lim_{(x,y)\to(x_0,y_0)} U(x,y) = U_0$, and $\lim_{(x,y)\to(x_0,y_0)} V(x,y) = V_0$.

Theorem 2.16.2 (continued 1)

Proof (continued). We now have
$\lim_{z\to z_0}(f(z)+F(z))$

$$\begin{aligned}
&= \lim_{(x,y)\to(x_0,y_0)} (u(x,y)+iv(x,0)+U(x,y)+iV(x,y)) \\
&= \lim_{(x,y)\to(x_0,y_0)}(u(x,y)+U(x,y)) + i\lim_{(x,y)\to(x_0,y_0)}(v(x,y)+V(x,y)) \\
&\qquad \text{by Theorem 2.16.1} \\
&= \lim_{(x,y)\to(x_0,y_0)} u(x,y) + \lim_{(x,y)\to(x_0,y_0)} U(x,y) \\
&\quad +i\left(\lim_{(x,y)\to(x_0,y_0)} v(x,y) + \lim_{(x,y)\to(x_0,y_0)} V(x,y)\right) \\
&\qquad \text{by the Sum Rule of Thomas' Theorem 1} \\
&= u_0+U_0+i(v_0+V_0) = u_0+iv_0+U_0+iV_0 = w_0+W_0,
\end{aligned}$$

and the first claim holds.

Theorem 2.16.2 (continued 2)

Proof (continued). Second,
$\lim_{z\to z_0}(f(z)F(z))$

$$\begin{aligned}
&= \lim_{(x,y)\to(x_0,y_0)} ((u(x,y)+iv(x,y))(U(x,y)+iV(x,y)) \\
&= \lim_{(x,y)\to(x_0,y_0)} (u(x,y)U(x,y)-v(x,y)V(x,y)) \\
&\quad +i \lim_{(x,y)\to(x_0,y_0)} (u(x,y)V(x,y)+v(x,y)U(x,y)) \text{ by Theorem 2.16.1} \\
&= \lim_{(x,y)\to(x_0,y_0)} u(x,y) \lim_{(x,y)\to(x_0,y_0)} U(x,y) \\
&\quad - \lim_{(x,y)\to(x_0,y_0)} v(x,y) \lim_{(x,y)\to(x_0,y_0)} V(x,y) \\
&\quad +i \left(\lim_{(x,y)\to(x_0,y_0)} u(x,y) \lim_{(x,y)\to(x_0,y_0)} V(x,y) \right. \\
&\quad \left. + \lim_{(x,y)\to(x_0,y_0)} v(x,y) \lim_{(x,y)\to(x_0,y_0)} U(x,y) \right)
\end{aligned}$$

Theorem 2.16.2 (continued 3)

Proof (continued).

$$\text{by the Sum Rule, Difference Rule, and Product Rule of Thomas' Theorem 1}$$
$$= u_0U_0 - v_0V_0 + i(u_0V_0 + v_0U_0) = (u_0 + iv_0)(U_0 + iV_0) = w_0W_0,$$

and the second claim holds.

Thirdly, suppose $W_0 \neq 0$. Then $U_0 \neq 0 \neq V_0$ and we have $\lim_{z\to z_0} f(z)/F(z)$

$$= \lim_{(x,y)\to(x_0,y_0)} (u(x,y) + iv(x,y))/(U(x,y) + iV(x,y)$$
$$= \lim_{(x,y)\to(x_0,y_0)} \left(\frac{u(x,y)U(x,y) + v(x,y)V(x,y)}{U(x,y)^2 + V(x,y)^2} + i\frac{v(x,y)U(x,y) - u(x,y)V(x,y)}{U(x,y)^2 + V(x,y)^2} \right)$$

Theorem 2.16.2 (continued 3)

Proof (continued).

$$\begin{aligned} & \text{by the Sum Rule, Difference Rule, and Product Rule} \\ & \text{of Thomas' Theorem 1} \\ = \; & u_0 U_0 - v_0 V_0 + i(u_0 V_0 + v_0 U_0) = (u_0 + iv_0)(U_0 + iV_0) = w_0 W_0, \end{aligned}$$

and the second claim holds.
Thirdly, suppose $W_0 \neq 0$. Then $U_0 \neq 0 \neq V_0$ and we have $\lim_{z \to z_0} f(z)/F(z)$

$$\begin{aligned} = \; & \lim_{(x,y)\to(x_0,y_0)} (u(x,y) + iv(x,y))/(U(x,y) + iV(x,y) \\ = \; & \lim_{(x,y)\to(x_0,y_0)} \left(\frac{u(x,y)U(x,y) + v(x,y)V(x,y)}{U(x,y)^2 + V(x,y)^2} \right. \\ & \left. + i \frac{v(x,y)U(x,y) - u(x,y)V(x,y)}{U(x,y)^2 + V(x,y)^2} \right) \end{aligned}$$

Theorem 2.16.2 (continued 4)

Proof (continued).

$$\begin{aligned}
&= \lim_{(x,y)\to(x_0,y_0)} \left(\frac{u(x,y)U(x,y) + v(x,y)V(x,y)}{U(x,y)^2 + V(x,y)^2} \right) \\
&\quad +i \lim_{(x,y)\to(x_0,y_0)} \left(\frac{v(x,y)U(x,y) - u(x,y)V(x,y)}{U(x,y)^2 + V(x,y)^2} \right) \\
&\qquad \text{by Theorem 2.16.1} \\
&= \left\{ \lim_{(x,y)\to(x_0,y_0)} u(x,y) \lim_{(x,y)\to(x_0,y_0)} U(x,y) \right. \\
&\quad \left. + \lim_{(x,y)\to(x_0,y_0)} v(x,y) \lim_{(x,y)\to(x_0,y_0)} V(x,y) \right\} \\
&\quad \Big/ \left(\lim_{(x,y)\to(x_0,y_0)} (U(x,y))^2 + \lim_{(x,y)\to(x_0,y_0)} (V(x,y))^2 \right)
\end{aligned}$$

Theorem 2.16.2 (continued 5)

Proof (continued).

$$+i\left\{\lim_{(x,y)\to(x_0,y_0)} v(x,y) \lim_{(x,y)\to(x_0,y_0)} U(x,y)\right.$$
$$\left. - \lim_{(x,y)\to(x_0,y_0)} u(x,y) \lim_{(x,y)\to(x_0,y_0)} V(x,y)\right\}$$
$$\bigg/ \left(\lim_{(x,y)\to(x_0,y_0)} (U(x,y))^2 + \lim_{(x,y)\to(x_0,y_0)} (V(x,y))^2\right)$$

by the Sum Rule, Difference Rule, Product Rule, Quotient Rule, and Power Rule of Thomas' Theorem 1

$$= \frac{u_0 U_0 + v_0 V_0}{U_0^2 + V_0^2} + i\frac{v_0 U_0 - u_0 V_0}{U_0^2 + V_0^2}$$
$$= (u_0 + iv_0)/(U_0 + iV_0) = w_0/W_0,$$

and the third claim holds. □

Lemma 2.16.A

Lemma 2.16.A. For any $z_0, c \in \mathbb{C}$, we have $\lim_{z \to z_0} c = c$ and $\lim_{z \to z_0} z = z_0$.

Proof. Let $f(z) = c$ and $g(z) = z$. Then f and g are defined in the entire complex plane $\mathbb{C}$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $0 < |z - z_0| < \delta$, then we have both $|f(z) - c| = |c - c| = 0 < \varepsilon$ and $|g(z) - z_0| = |z - z_0| < \delta = \varepsilon$. So $\lim_{z \to z_0} c = c$ and $\lim_{z \to z_0} z = z_0$. □

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Proof. Let $f(z) = c$ and $g(z) = z$. Then f and g are defined in the entire complex plane $\mathbb{C}$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $0 < |z - z_0| < \delta$, then we have both $|f(z) - c| = |c - c| = 0 < \varepsilon$ and $|g(z) - z_0| = |z - z_0| < \delta = \varepsilon$. So $\lim_{z \to z_0} c = c$ and $\lim_{z \to z_0} z = z_0$. □

Lemma 2.16.B

Lemma 2.16.B. For any $z_0 \in \mathbb{C}$ and $n \in \mathbb{N}$, we have $\lim_{z \to z_0} z^n = z_0^n$.

Proof. Let $f(z) = z^n$. Then f is defined in the entire complex plane $\mathbb{C}$ and

$$\begin{aligned} \lim_{z \to z_0} f(z) &= \lim_{z \to z_0} z^n \\ &= \underbrace{\left(\lim_{z \to z_0} z\right)\left(\lim_{z \to z_0} z\right)\cdots\left(\lim_{z \to z_0} z\right)}_{n \text{ times}} \text{ by Theorem 2.16.2} \\ &\quad \text{(2nd claim) and mathematical induction} \\ &= (z_0)(z_0)\cdots(z_0) \text{ by Lemma 2.16.A} \\ &= z_0^n. \end{aligned}$$

□

Lemma 2.16.B

Lemma 2.16.B. For any $z_0 \in \mathbb{C}$ and $n \in \mathbb{N}$, we have $\lim_{z \to z_0} z^n = z_0^n$.

Proof. Let $f(z) = z^n$. Then f is defined in the entire complex plane $\mathbb{C}$ and

$$\begin{aligned}
\lim_{z \to z_0} f(z) &= \lim_{z \to z_0} z^n \\
&= \underbrace{\left(\lim_{z \to z_0} z\right)\left(\lim_{z \to z_0} z\right)\cdots\left(\lim_{z \to z_0} z\right)}_{n \text{ times}} \text{ by Theorem 2.16.2} \\
&\quad \text{(2nd claim) and mathematical induction} \\
&= (z_0)(z_0)\cdots(z_0) \text{ by Lemma 2.16.A} \\
&= z_0^n.
\end{aligned}$$

□

Corollary 2.16.A

Corollary 2.16.A. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree n. Then $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof. First, P is defined in the entire complex plane $\mathbb{C}$. We have

$$\begin{aligned}
\lim_{z \to z_0} P(z) &= \lim_{z \to z_0} (a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n) \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} (a_1 z) + \lim_{z \to z_0} (a_2 z^2) + \cdots + \lim_{z \to z_0} (a_n z^n) \\
&\quad \text{by Theorem 2.16.2 (first claim) and induction} \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} a_1 \lim_{z \to z_0} z + \lim_{z \to z_0} a_2 \lim_{z \to z_0} (z^2) + \cdots \\
&\quad + \lim_{z \to z_0} a_n \lim_{z \to z_0} (z^n) \text{ by Theorem 2.16.2 (second claim)} \\
&= a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_0^n \text{ by Lemmas 2.16.A \& 2.16.B} \\
&= P(z_0).
\end{aligned}$$

□

Corollary 2.16.A

Corollary 2.16.A. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree n. Then $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof. First, P is defined in the entire complex plane $\mathbb{C}$. We have

$$\begin{aligned}
\lim_{z \to z_0} P(z) &= \lim_{z \to z_0} (a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n) \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} (a_1 z) + \lim_{z \to z_0} (a_2 z^2) + \cdots + \lim_{z \to z_0} (a_n z^n) \\
&\quad \text{by Theorem 2.16.2 (first claim) and induction} \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} a_1 \lim_{z \to z_0} z + \lim_{z \to z_0} a_2 \lim_{z \to z_0} (z^2) + \cdots \\
&\quad + \lim_{z \to z_0} a_n \lim_{z \to z_0} (z^n) \text{ by Theorem 2.16.2 (second claim)} \\
&= a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_0^n \text{ by Lemmas 2.16.A \& 2.16.B} \\
&= P(z_0).
\end{aligned}$$

□

Corollary 2.16.B

Corollary 2.16.B. Let $R(z) = P_1(z)/P_2(z)$ be a rational function; that is, R is the quotient of polynomials P_1 and P_2. Then $\lim_{z\to z_0} R(z) = R(z_0)$, provided $P_2(z_0) \neq 0$.

Proof. By Corollary 2.16.A, $\lim_{z\to z_0} P_1(z) = P_1(z_0)$ and $\lim_{z\to z_0} P_2(z) = P_2(z_0) \neq 0$. By Theorem 2.16.2 (third claim)

$$\lim_{z\to z_0} R(z) = \lim_{z\to z_0}(P_1(z)/P_2(z)) = \left(\lim_{z\to z_0} P_1(z)\right)\Big/\left(\lim_{z\to z_0} P_2(z)\right)$$

$$= P_1(z_0)/P_2(z_0) = R(z_0).$$

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$$\lim_{z\to z_0} R(z) = \lim_{z\to z_0} (P_1(z)/P_2(z)) = \left(\lim_{z\to z_0} P_1(z)\right) \Big/ \left(\lim_{z\to z_0} P_2(z)\right)$$

$$= P_1(z_0)/P_2(z_0) = R(z_0).$$

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