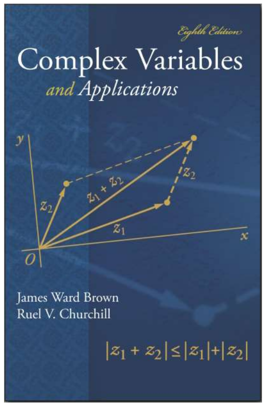
Complex Variables

Chapter 2. Analytic Functions

Section 2.17. Limits Involving the Point at Infinity—Proofs of Theorems



Theorem 2.17.1

Theorem 2.17.1. If $z_0, w_0 \in \mathbb{C}$ then

$$\lim_{z\to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to z_0} 1/f(z) = 0$$
$$\lim_{z\to\infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z\to 0} f(1/z) = w_0, \text{ and}$$
$$\lim_{z\to\infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to 0} 1/f(1/z) = 0.$$

Proof. Let $\varepsilon > 0$ and define $g(z) = 1/f(z)$, $h(z) = f(1/z)$, and $k(z) = 1/f(1/z)$.
First, suppose $\lim_{z\to z_0} f(z) = \infty$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $1/|f(z)| < \varepsilon$. So $0 < |z - z_0| < \delta$ implies $1/|f(z)| = |g(z) - 0| < \varepsilon$. Therefore (by definition) $\lim_{z\to z_0} g(z) = \lim_{z\to z_0} 1/f(z) = 0$.
Next, suppose $\lim_{z\to z_0} 1/f(z) = 0$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|1/f(z) - 0| < \varepsilon$. So $0 < |z - z_0| < \delta$ implies $|1/f(z) - 0| = 1/|f(z)| < \varepsilon$. Therefore (by definition) $\lim_{z\to z_0} f(z) = \infty$.

Theorem 2.17.1 (continued 1)

Proof (continued). Second, suppose $\lim_{z\to\infty} f(z) = w_0$. Then (by definition) there exists $\delta > 0$ such that $1/|z| < \delta$ implies $|f(z) - w_0| < \varepsilon$. So (replacing z with $1/z$) $0 < |z| = |z - 0| < \delta$ implies $|f(1/z) - w_0| = |h(z) - w_0| < \varepsilon$. Therefore (by definition) $\lim_{z\to 0} h(z) = \lim_{z\to 0} f(1/z) = w_0$.

Suppose $\lim_{z\to 0} f(1/z) = w_0$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|f(1/z) - w_0| < \varepsilon$. So (replacing z with $1/z$) $0 < |1/z - 0| = 1/|z| < \delta$ implies $|f(z) - w_0| < \varepsilon$. Therefore (by definition) $\lim_{z\to\infty} f(z) = w_0$.

Third, suppose $\lim_{z\to\infty} f(z) = \infty$. Then (by definition) there exists $\delta > 0$ such that $1/|z| < \delta$ implies $1/|f(z)| < \varepsilon$. So (replacing z with $1/z$) $0 < |z| < \delta$ implies $|1/f(1/z)| < \varepsilon$. So $0 < |z - 0| < \delta$ implies $|k(z) - 0| < \varepsilon$. Therefore (by definition) $\lim_{z\to 0} k(z) = \lim_{z\to 0} 1/f(1/z) = 0$.

Theorem 2.17.1 (continued 2)

Theorem 2.17.1. If $z_0, w_0 \in \mathbb{C}$ then

$$\lim_{z\to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to z_0} 1/f(z) = 0$$
$$\lim_{z\to\infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z\to 0} f(1/z) = w_0, \text{ and}$$
$$\lim_{z\to\infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to 0} 1/f(1/z) = 0.$$

Proof (continued). Suppose $\lim_{z\to 0} 1/f(1/z) = 0$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|1/f(1/z) - 0| < \varepsilon$. So (replacing z with $1/z$) $0 < |1/z| < \delta$ implies $|1/f(z)| < \varepsilon$. Therefore (by definition) $\lim_{z\to\infty} f(z) = \infty$. □