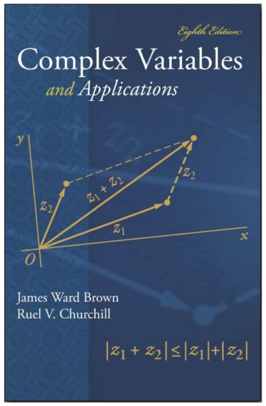
Complex Variables

Chapter 2. Analytic Functions

Section 2.18. Continuity—Proofs of Theorems



Theorem 2.18.1

Theorem 2.18.1. If f is continuous at z_0 and g is continuous at $f(z_0)$ (so z_0 is an interior or boundary point of the domain of f and $f(z_0)$ is an interior or boundary point of the domain of g), then $(g \circ f)(z) = g(f(z))$ is continuous at z_0.

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(z_0)$, then $\lim_{w \to f(z_0)} g(w) = g(f(z_0))$, so there is $\delta_1 > 0$ such that $|w - f(z_0)| < \delta_1$ (and w is in the domain of g) implies $|g(w) - g(f(z_0))| < \varepsilon$. Since f is continuous at z_0 then $\lim_{z \to z_0} f(z) = f(z_0)$, so there is $\delta_2 > 0$ such that $|z - z_0| < \delta_2$ (and z is in the domain of f) implies $|f(z) - f(z_0)| < \delta_1$. So if z is in the domain of $g \circ f$ and $|z - z_0| < \delta_2$ then (with $w = f(z)$) $|w - f(z_0)| = |f(z) - f(z_0)| < \delta_1$ and so $|g(w) - g(f(z_0))| = |g(f(z)) - g(f(z_0))| < \varepsilon$. Therefore, $\lim_{z \to z_0} g(f(z)) = g(f(z_0))$. So $g \circ f$ is continuous at z_0, as claimed. □

Theorem 2.18.2

Theorem 2.18.2. If f is continuous at z_0 (an interior or boundary point of the domain of f) and $f(z_0) \neq 0$ then $f(z) \neq 0$ throughout some neighborhood of z_0.

Proof. Let $\varepsilon = |f(z_0)|/2 > 0$. Since f is continuous at z_0 then there is $\delta > 0$ such that if $|z - z_0| < \delta$ (and z is in the domain of f) then $|f(z) - f(z_0)| < \varepsilon = |f(z_0)|/2$. ASSUME $f(z') = 0$ for some z' with $|z' - z_0| < \delta$. Then $|f(z') - f(z_0)| < |f(z_0)|/2$ implies that $|f(z_0)| < |f(z_0)|/2$, a CONTRADICTION. So the assumption that $f(z') = 0$ for some z' with $|z' - z_0| < \delta$ is false, and so for all z in the domain of f with $|z - z_0| < \delta$ we have $f(z) \neq 0$. That is, there is a neighborhood of z_0 on which $f(z) \neq 0$, as claimed. □