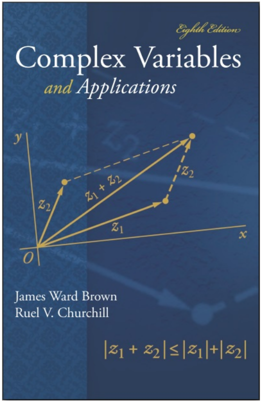
Complex Variables

Chapter 2. Analytic Functions

Section 2.18. Continuity—Proofs of Theorems



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Proof. Let $\varepsilon > 0$. Since g is continuous at $f(z_0)$, then $\lim_{w \to f(z_0)} g(w) = g(f(z_0))$, so there is $\delta_1 > 0$ such that $|w - f(z_0)| < \delta_1$ (and w is in the domain of g) implies $|g(w) - g(f(z_0))| < \varepsilon$.

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