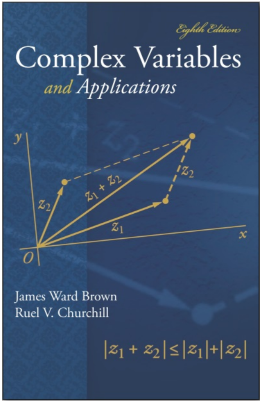
Complex Variables

Chapter 2. Analytic Functions

Section 2.19. Derivatives—Proofs of Theorems



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Theorem 2.19.A

Theorem 2.19.A. Differentiable implies Continuous. If f is differentiable at point z_0 then f is continuous at z_0.

Proof. First, by the definition of derivative, the domain of f contains some open neighborhood of z_0. Also,

$$\begin{aligned}
\lim_{z\to z_0}(f(z)-f(z_0)) &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)\\
&= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0) \text{ provided each constituent limit exists}\\
&= f'(z_0)\cdot 0 = 0.
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Therefore $\lim_{z\to z_0} f(z) - \lim_{z\to z_0} f(z_0) = 0$ or $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} f(z_0) = f(z_0)$. So, by definition, f is continuous at z_0. □

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