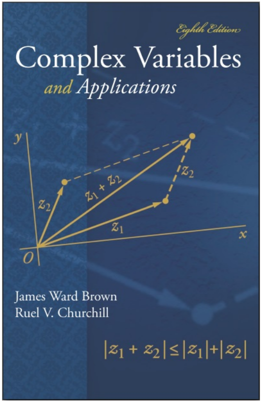
Complex Variables

Chapter 2. Analytic Functions

Section 2.20. Differentiation Formulas—Proofs of Theorems



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Theorem 2.20.A

Theorem 2.20.A. Let $c \in \mathbb{C}$ and let f and g be functions where derivatives exist at a point $z \in \mathbb{C}$. Then:

$$\frac{d}{dz}[c] = 0, \ \frac{d}{dz}[z] = 1, \ \frac{d}{dz}[cf(z)] = c\frac{d}{dz}[f],$$

$$\text{and } \frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)].$$

Proof. By the definition of derivative:

$$\begin{aligned}
\frac{d}{dz}[c] &= \lim_{\Delta z \to 0} \frac{(c) - (c)}{\Delta z} = 0 \\
\frac{d}{dz}[z] &= \lim_{\Delta z \to 0} \frac{(z + \Delta z) - z}{\Delta z} = 1 \\
\frac{d}{dz}[cf(z)] &= \lim_{\Delta z \to 0} \frac{cf(z + \Delta z) - cf(z)}{\Delta z} \\
&= c \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = c\frac{d}{dz}[f(z)]
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Theorem 2.20.A (continued)

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$$\text{and } \frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)].$$

Proof (continued). By the definition of derivative:

$$\begin{aligned}
\frac{d}{dz}[f(z) + g(z)] &= \lim_{\Delta z \to 0} \frac{(f(z+\Delta z) + g(z+\Delta z) - (f(z) + g(z))}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} \\
&= \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)] \qquad \square
\end{aligned}$$

Theorem 2.20.B

Theorem 2.20.B. If f and g are function whose derivative exists at a point z then

$$\text{Product Rule: } \frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z) \text{ and}$$

$$\text{Quotient Rule: } \frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2} \text{ if } g(z) \neq 0.$$

Proof. By the definition of derivative, $\frac{d}{dz}[f(z)g(z)] =$

$$\begin{aligned} &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} (f(z+\Delta z)g(z+\Delta z) - f(z)g(z+\Delta z) + f(z)g(z+\Delta z) \\ &\qquad -f(z)g(z))/\Delta z \\ &= \lim_{\Delta z \to 0} \left(\frac{(f(z+\Delta z) - f(z))g(z+\Delta z)}{\Delta z} + \frac{f(z)(g(z+\Delta z) - g(z))}{\Delta z} \right) \end{aligned}$$

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&= \lim_{\Delta z \to 0} \left(\frac{(f(z+\Delta z) - f(z))g(z+\Delta z)}{\Delta z} + \frac{f(z)(g(z+\Delta z) - g(z))}{\Delta z} \right)
\end{aligned}$$

Theorem 2.20.B (continued)

Proof (continued).

$$\begin{aligned} &= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}\lim_{\Delta z\to 0} g(z+\Delta z) \\ &\qquad + \lim_{\Delta z\to 0} f(z)\lim_{\Delta z\to 0}\frac{g(z+\Delta z)-g(z)}{\Delta z} \\ &= f'(z)\lim_{\Delta z\to 0} g(z+\Delta z)+f(z)g'(z) \\ &= f'(z)g(z)+f(z)g'(z) \text{ since } g \text{ is continuous at } z \text{ by Theorem 2.19.A.} \end{aligned}$$

By the definition of derivative, $\dfrac{d}{dz}[f(z)/g(z)] =$

$$\begin{aligned} \frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] &= \lim_{\Delta z\to 0}\frac{\frac{f(z+\Delta z)}{g(z+\Delta z)}-\frac{f(z)}{g(z)}}{\Delta z} \\ &= \lim_{\Delta z\to 0}\frac{g(z)f(z+\Delta z)-f(z)g(z+\Delta z)}{\Delta z g(z+\Delta z)g(z)} \end{aligned}$$

Theorem 2.20.B (continued)

Proof (continued).

$$\begin{aligned}
&= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \lim_{\Delta z \to 0} g(z+\Delta z) \\
&\qquad + \lim_{\Delta z \to 0} f(z) \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} \\
&= f'(z) \lim_{\Delta z \to 0} g(z+\Delta z) + f(z)g'(z) \\
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&= \lim_{\Delta z \to 0} \frac{g(z)f(z+\Delta z) - g(z)f(z) + g(z)f(z) - f(z)g(z+\Delta z)}{\Delta z g(z+\Delta z)g(z)} \\
&= \lim_{\Delta z \to 0} \frac{g(z)\frac{f(z+\Delta z)-f(z)}{\Delta z} - f(z)\frac{g(z+\Delta z)-g(z)}{\Delta z}}{g(z+\Delta z)g(z)} \\
&= \frac{\lim_{\Delta z \to 0} g(z)\frac{f(z+\Delta z)-f(z)}{\Delta z} - \lim_{\Delta z \to 0} f(z)\frac{g(z+\Delta z)-g(z)}{\Delta z}}{\lim_{\Delta z \to 0} g(z+\Delta z)g(z)} \\
&= \frac{g(z)\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} - f(z)\lim_{\Delta z \to 0} \frac{g(z+\Delta z)-g(z)}{\Delta z}}{g(z)\lim_{\Delta z \to 0} g(z+\Delta z)} \\
&= \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2} = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.
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Corollary 2.20.A

Corollary 2.20.A. If $n \in \mathbb{N}$ then $\frac{d}{dz}[z^n] = nz^{n-1}$.

Proof. The traditional way to prove this is using the Binomial Theorem (Theorem 1.3.2) and you are asked to do this in Exercise 2.20.6b. Here we give an inductive proof proof (this is Exercise 2.20.6a).

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We have the result $\frac{d}{dz}[z^1] = 1$ by Theorem 2.20.A so the base case $n = 1$ holds.

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We have the result $\frac{d}{dz}[z^1] = 1$ by Theorem 2.20.A so the base case $n = 1$ holds. Now assume the result holds for $n = k$ and consider $n = k + 1$:

$$\begin{aligned} \frac{d}{dz}[z^n] &= \frac{d}{dz}[z^{k+1}] = \frac{d}{dz}[z^k z] \\ &= [kz^{k-1}](z) + (z^k)[1] \text{ by the Product Rule (Theorem 2.20.B)} \\ &\quad \text{and the induction hypothesis} \\ &= kz^k + z^k = (k+1)z^k. \end{aligned}$$

Therefore the claim holds for $n = k + 1$ and hence by Mathematical Induction it holds for all $n \in \mathbb{N}$. □

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Theorem 2.20.C

Theorem 2.20.C. The Chain Rule.
Suppose that f has a derivative at z_0 and that g has a derivative at $f(z_0)$. Then the composition function $F(z) = (g \circ f)(z) = g(f(z))$ has a derivative at z_0 and $F'(z_0) = g'(f(z_0))f'(z_0)$. In differential notation with $w = f(z)$ and $W = g(w)$ (so that $W = F(z)$) we have $\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}$.

Proof. Denote $w_0 = f(z_0)$. Since $g'(w_0) = g'(f(z_0))$ exists by hypothesis then by the definition of derivative, there is some $\varepsilon > 0$ such that g is defined on $|w - w_0| < \varepsilon$.

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$$\Phi(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} - g'(w_0) & \text{if } w \neq w_0 \\ 0 & \text{if } w = w_0 \end{cases}$$

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Theorem 2.20.C (continued 1)

Proof (continued). ...and note that

$$\lim_{w\to w_0}\Phi(w)=\lim_{w\to w_0}\left(\frac{g(w)-g(w_0)}{w-w_0}-g'(w_0)\right)$$

$$=\lim_{w\to w_0}\left(\frac{g(w)-g(w_0)}{w-w_0}\right)-g'(w_0)=g'(w_0)-g'(w_0)=0=\Phi(w_0).$$

So Φ is continuous at w_0. Rearranging the definition of Φ we get

$$g(w)-g(w_0)=(g'(w_0)+\Phi(w))(w-w_0) \text{ for } |w-w_0|<\varepsilon.$$

Since $f'(z_0)$ exists then f is continuous at z_0 by Theorem 2.19.A, so by definition of continuity there is $\delta>0$ such that for all $|z-z_0|<\delta$ we have $|f(z)-f(z_0)|=|w-w_0|<\varepsilon$.

Theorem 2.20.C (continued 1)

Proof (continued). ... and note that

$$\lim_{w\to w_0} \Phi(w) = \lim_{w\to w_0} \left(\frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \right)$$

$$= \lim_{w\to w_0} \left(\frac{g(w) - g(w_0)}{w - w_0} \right) - g'(w_0) = g'(w_0) - g'(w_0) = 0 = \Phi(w_0).$$

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$$g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z)))(f(z) - f(z_0)) \text{ for } |z - z_0| < \delta$$

Theorem 2.20.C (continued 1)

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$$g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z)))(f(z) - f(z_0)) \text{ for } |z - z_0| < \delta$$

Theorem 2.20.C (continued 2)

Proof (continued). . . . or

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = (g'(f(z_0)) + \Phi(f(z)) \frac{f(z) - f(z_0)}{z - z_0} \text{ for } 0 < |z - z_0| < \delta.$$

Therefore

$$\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} (g'(f(z_0) + \Phi(f(z))) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

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or

$$\begin{aligned} \frac{d}{dz}[g(f(z))]\Big|_{z=z_0} &= g'(f(z_0))f'(z_0) + \lim_{z \to z_0} \Phi(f(z))f'(z_0) \\ &= g'(f(z_0))f'(z_0) + \Phi(f(z_0))f'(z_0) \text{ since } f \text{ is} \\ &\quad \text{continuous at } z_0 \text{ and } \Phi \text{ is continuous at } w_0 = f(z_0) \\ &= g'(f(z_0))f'(z_0) \text{ since } \Phi(f(z_0)) = \Phi(w_0) = 0. \end{aligned}$$

□

Theorem 2.20.C (continued 2)

Proof (continued). . . . or

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