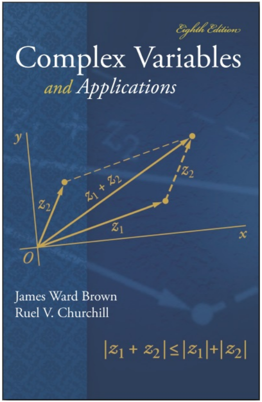
Complex Variables

Chapter 2. Analytic Functions

Section 2.21. Cauchy-Riemann Equations—Proofs of Theorems



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Theorem 2.21.A

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Suppose that $f(z) = u(x, y) + iv(x, y)$ and that f' exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations:

$$\frac{\partial}{\partial x}[u(x, y)] = \frac{\partial}{\partial y}[v(x, y)] \text{ and } \frac{\partial}{\partial y}[u(x, y)] = -\frac{\partial}{\partial x}[v(x, y)]$$

(or with subscripts representing partial derivatives, $u_x = v_y$ and $u_y = -v_x$) at (x_0, y_0). Also, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. Suppose f' exists at $z_0 = x_0 + iy_0$. Let $\Delta z = \Delta x + i\Delta y$. Then with $w = f(z)$ we have:

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Proof. Suppose f' exists at $z_0 = x_0 + iy_0$. Let $\Delta z = \Delta x + i\Delta y$. Then with $w = f(z)$ we have:

Theorem 2.21.A (continued 1)

Proof (continued).

$$\frac{\Delta w}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{\{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\} + i\{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\}}{\Delta x + i\Delta y}$$

Then $f'(z_0) = \lim_{\Delta z \to 0}(\Delta w/\Delta z)$, so by Theorem 2.16.1,

$$f'(z_0) = \lim_{(\Delta x, \Delta y)\to(0,0)} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x, \Delta y)\to(0,0)} \operatorname{Im}\left(\frac{\Delta w}{\Delta z}\right). \quad (3)$$

We now apply the contrapositive of the Two-Path Test for the Nonexistence of a Limit for a function of two variables (see Note 2.15.A), which implies that if a limit exists as $\Delta z \to 0$ then the limit exists and is the same along all paths for which $\Delta z \to 0$.

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Then $f'(z_0) = \lim_{\Delta z \to 0}(\Delta w/\Delta z)$, so by Theorem 2.16.1,

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \text{Re}\left(\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \text{Im}\left(\frac{\Delta w}{\Delta z}\right). \qquad (3)$$

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Theorem 2.21.A (continued 2)

Proof (continued). In particular, we can let $\Delta z \to 0$ along the real axis (where $\Delta y = 0$) or along the imaginary axis (where $\Delta x = 0$).

We first consider $\Delta z \to 0$ along the real axis and have for the real and imaginary parts of equation (3) that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \text{Re}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0)$$

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$$= \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} = v_x(x_0,y_0).$$

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So by (3), $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$ and f' has the form as claimed.

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Proof (continued). Second, with $\Delta z \to 0$ along the imaginary axis so that $\Delta z = i\Delta y$, $\Delta y \to 0$, and $\Delta x = 0$, we have

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$$

and so

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} = v_y(x_0, y_0)$$

and

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Proof (continued). Since
$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$, then we must have $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $v_x(x_0, y_0) = -u_y(x_0, y_0)$. □