Complex Variables

Chapter 2. Analytic Functions

Section 2.22. Sufficient Conditions for Differentiability—Proofs of Theorems

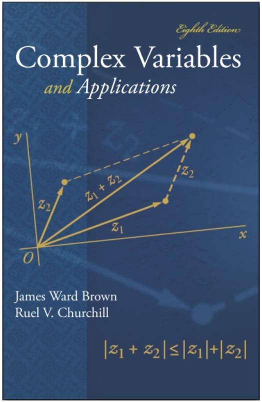

Table of contents

Theorem 2.22.A

Theorem 2.22.A. The Cauchy-Riemann Equations and Continuity Imply Differentiability

Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $y_y(x_0, y_0) = -v_x(x_0, y_0)$.

Then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. We present the proof given by Brown and Churchill. A more self-contained proof based on the Mean Value Theorem is given in my notes for Complex Analysis 1 (MATH 5510) on III.2. Analytic Functions.

Theorem 2.22.A

Theorem 2.22.A. The Cauchy-Riemann Equations and Continuity Imply Differentiability

Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $y_y(x_0, y_0) = -v_x(x_0, y_0)$.

Then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. We present the proof given by Brown and Churchill. A more self-contained proof based on the Mean Value Theorem is given in my notes for Complex Analysis 1 (MATH 5510) on III.2. Analytic Functions.

Theorem 2.22A (continued 1)

Proof (continued). Let $\Delta z = \Delta x + i\Delta y$ where $0 < |\Delta z| < \varepsilon$ and let $\Delta w = f(z_0 + \Delta z) - f(z_0)$. We take $\Delta w = \Delta u + i\Delta v$ where

$$\Delta u = u(x_0+\Delta x, y_0+\Delta y_0)-u(x_0, y_0), \;\; \Delta v = v(x_0+\Delta x, y_0+\Delta y_0)-v(x_0, y_0).$$

By (b), the first order partial derivatives of u and v are continuous at (x_0, y_0), so by a result from advanced calculus (see W. Kaplan's *Advanced Calculus*, 5th ed., page 86 (2003)) we may write

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y$$

where $\varepsilon_i \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ for $i = 1, 2, 3, 4$.

Theorem 2.22A (continued 1)

Proof (continued). Let $\Delta z = \Delta x + i\Delta y$ where $0 < |\Delta z| < \varepsilon$ and let $\Delta w = f(z_0 + \Delta z) - f(z_0)$. We take $\Delta w = \Delta u + i\Delta v$ where

$$\Delta u = u(x_0+\Delta x, y_0+\Delta y_0)-u(x_0, y_0), \;\; \Delta v = v(x_0+\Delta x, y_0+\Delta y_0)-v(x_0, y_0).$$

By (b), the first order partial derivatives of u and v are continuous at (x_0, y_0), so by a result from advanced calculus (see W. Kaplan's *Advanced Calculus*, 5th ed., page 86 (2003)) we may write

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y$$

where $\varepsilon_i \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ for $i = 1, 2, 3, 4$. So we can express

$$\Delta w = (u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y)$$

$$+i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y).$$

Theorem 2.22A (continued 1)

Proof (continued). Let $\Delta z = \Delta x + i\Delta y$ where $0 < |\Delta z| < \varepsilon$ and let $\Delta w = f(z_0 + \Delta z) - f(z_0)$. We take $\Delta w = \Delta u + i\Delta v$ where

$$\Delta u = u(x_0+\Delta x, y_0+\Delta y_0)-u(x_0, y_0),\ \ \Delta v = v(x_0+\Delta x, y_0+\Delta y_0)-v(x_0, y_0).$$

By (b), the first order partial derivatives of u and v are continuous at (x_0, y_0), so by a result from advanced calculus (see W. Kaplan's *Advanced Calculus*, 5th ed., page 86 (2003)) we may write

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y$$

where $\varepsilon_i \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ for $i = 1, 2, 3, 4$. So we can express

$$\Delta w = (u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y)$$

$$+i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y).$$

Theorem 2.22.A (continued 2)

Proof (continued). Since we hypothesize that the Cauchy-Riemann equations are satisfied at (x_0, y_0), then $u_y(x_0, y_0) = -v_x(x_0, y_0)$ and $v_y(x_0, y_0) = u_x(x_0, y_0)$ and so

$$\Delta w = (u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y)$$

$$+ i(v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y)$$

$$= u_x(x_0, y_0)(\Delta x + i\Delta y) + iv_x(x_0, y_0)(\Delta x + i\Delta y) + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y$$

and

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}.$$

But $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$ (by the Triangle Inequality, say), so $|\Delta x/\Delta z| \le 1$ and $|\Delta y/\Delta z| \le 1$.

Theorem 2.22.A (continued 2)

Proof (continued). Since we hypothesize that the Cauchy-Riemann equations are satisfied at (x_0, y_0), then $u_y(x_0, y_0) = -v_x(x_0, y_0)$ and $v_y(x_0, y_0) = u_x(x_0, y_0)$ and so

$$\Delta w = (u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y)$$

$$+i(v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y)$$

$$= u_x(x_0, y_0)(\Delta x + i\Delta y) + iv_x(x_0, y_0)(\Delta x + i\Delta y) + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y$$

and

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}.$$

But $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$ (by the Triangle Inequality, say), so $|\Delta x/\Delta z| \leq 1$ and $|\Delta y/\Delta z| \leq 1$.

Theorem 2.22.A (continued 3)

Proof (continued). Consequently,

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| = |\varepsilon_1 + i\varepsilon_3|\left|\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_1 + i\varepsilon_3| \le |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta x}{\Delta z}\right| = |\varepsilon_2 + i\varepsilon_4|\left|\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_2| + |\varepsilon_4|.$$

So as $\Delta z = \Delta z + i\Delta y \to 0$, we have that $|(\varepsilon_1 + i\varepsilon_3)\Delta x/\Delta z| \to 0$ and $|(\varepsilon_2 + i\varepsilon_4)\Delta x/\Delta z| \to 0$. Therefore,

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \to 0}\left(u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right) \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0). \end{aligned}$$

□

Theorem 2.22.A (continued 3)

Proof (continued). Consequently,

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| = |\varepsilon_1 + i\varepsilon_3|\left|\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_1 + i\varepsilon_3| \le |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta x}{\Delta z}\right| = |\varepsilon_2 + i\varepsilon_4|\left|\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_2| + |\varepsilon_4|.$$

So as $\Delta z = \Delta z + i\Delta y \to 0$, we have that $|(\varepsilon_1 + i\varepsilon_3)\Delta x/\Delta z| \to 0$ and $|(\varepsilon_2 + i\varepsilon_4)\Delta x/\Delta z| \to 0$. Therefore,

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(u_x(x_0, y_0) + i v_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z} \right) \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0). \end{aligned}$$

$\square$