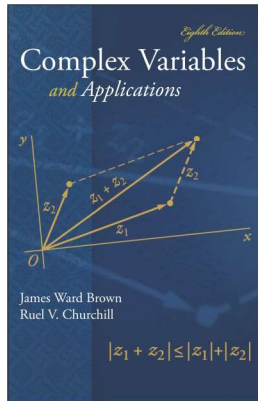


# Complex Variables

## Chapter 2. Analytic Functions

### Section 2.23. Polar Coordinates—Proofs of Theorems



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Lemma 2.23.A.

## Lemma 2.23.A

**Lemma 2.23.A.** Let the function  $f(z) = u(x, y) + iv(x, y)$  be defined throughout some  $\varepsilon$  neighborhood of a point  $z_0 = x_0 + iy_0$ , and suppose that

- the first-order partial derivatives of the functions  $u$  and  $v$  with respect to  $x$  and  $y$  exist everywhere in the neighborhood, and
- those partial derivatives are continuous at  $(x_0, y_0)$  and satisfy the Cauchy-Riemann equations  $u_x(x_0, y_0) = v_y(x_0, y_0)$  and  $y_v(x_0, y_0) = -v_x(x_0, y_0)$ .

Then with  $z_0 = r_0 \exp(i\theta) \neq 0$  we have

$$r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \text{ and } u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0).$$

These are the polar coordinate forms of the Cauchy-Riemann equations.

**Proof.** We have  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  and, for  $z \neq 0$ ,  $z = r \exp(i\theta)$ . Also,  $x = r \cos \theta$  and  $y = r \sin \theta$ .

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Lemma 2.23.A.

## Lemma 2.23.A (continued)

**Proof (continued).** By the Chain Rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = u_x \cos \theta + u_y \sin \theta \text{ and}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -u_x r \sin \theta + u_y r \cos \theta. \quad (2)$$

$$\text{Similarly, } \frac{\partial v}{\partial r} = v_x \cos \theta + v_y \sin \theta \text{ and } \frac{\partial v}{\partial \theta} = -v_x r \sin \theta + v_y r \cos \theta. \quad (3)$$

Assuming the Cauchy-Riemann equations in  $(x, y)$  hold, we have  $u_x = v_y$  and  $u_y = -v_x$  at  $(x_0, y_0)$ . So from (5)

$$v_r = v_x \cos \theta + v_y \sin \theta = -u_y \cos \theta + u_x \sin \theta \text{ and}$$

$$v_\theta = -v_x r \sin \theta + v_y r \cos \theta = u_y r \sin \theta + u_x r \cos \theta \quad (5)$$

at  $(r_0, \theta_0)$ . Comparing (2) and (5) we have  $ru_r = v_\theta$  and  $u_\theta = -rv_r$  at  $(r_0, \theta_0)$ .  $\square$

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