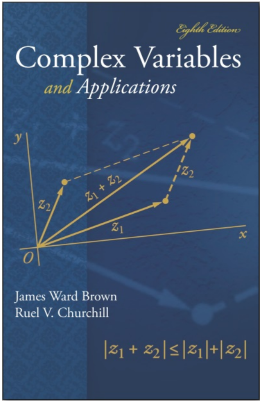
Complex Variables

Chapter 2. Analytic Functions

Section 2.23. Polar Coordinates—Proofs of Theorems



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Lemma 2.23.A

Lemma 2.23.A. Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $y_y(x_0, y_0) = -v_x(x_0, y_0)$.

Then with $z_0 = r_0 \exp(i\theta) \neq 0$ we have

$$r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \text{ and } u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0).$$

These are the polar coordinate forms of the Cauchy-Riemann equations.

Proof. We have $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and, for $z \neq 0$, $z = r \exp(i\theta)$. Also, $x = r \cos\theta$ and $y = r \sin\theta$.

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Lemma 2.23.A (continued)

Proof (continued). By the Chain Rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta \text{ and}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta. \qquad (2)$$

Similarly, $$\frac{\partial v}{\partial r} = v_x\cos\theta + v_y\sin\theta \text{ and } \frac{\partial v}{\partial \theta} = -v_x r\sin\theta + v_y r\cos\theta. \qquad (3)$$

Assuming the Cauchy-Riemann equations in (x, y) hold, we have $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0). So from (5)

$$v_r = v_x\cos\theta + v_y\sin\theta = -u_y\cos\theta + u_x\sin\theta \text{ and}$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta = u_y r\sin\theta + u_x r\cos\theta \qquad (5)$$

at (r_0, θ_0).

Lemma 2.23.A (continued)

Proof (continued). By the Chain Rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta \text{ and}$$

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Similarly, $$\frac{\partial v}{\partial r} = v_x \cos\theta + v_y \sin\theta \text{ and } \frac{\partial v}{\partial \theta} = -v_x r \sin\theta + v_y r \cos\theta. \qquad (3)$$

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$$v_\theta = -v_x r \sin\theta + v_y r \cos\theta = u_y r \sin\theta + u_x r \cos\theta \qquad (5)$$

at (r_0, θ_0). Comparing (2) and (5) we have $ru_r = v_\theta$ and $u_\theta = -rv_r$ at (r_0, θ_0). □

Lemma 2.23.A (continued)

Proof (continued). By the Chain Rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta \text{ and}$$

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Similarly, $\dfrac{\partial v}{\partial r} = v_x \cos\theta + v_y \sin\theta$ and $\dfrac{\partial v}{\partial \theta} = -v_x r \sin\theta + v_y r \cos\theta. \qquad (3)$

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