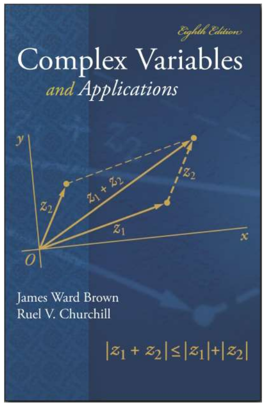
Complex Variables

Chapter 2. Analytic Functions

Section 2.24. Analytic Functions—Proofs of Theorems



Theorem 2.24.A

Theorem 2.24.A. If $f'(z) = 0$ everywhere in a domain D, then f must be constant throughout D.

Proof. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Since $f'(z) = 0$ for all $z \in D$ (where D is an open connected set), then f is differentiable on D and so satisfies the Cauchy-Riemann equations. By Theorem 2.21.A, $f'(z) = f'(x + iy) = u_x(x, y) + iv_x(x, y)$ and by the Cauchy-Riemann equations $f'(z) = f'(x + iy) = v_y(x, y) - iu_y(x, y)$. Since $f'(z) = 0$ in D, then $u_x(x, y) = u_y(x, y) = 0$ and $v_x(x, y) = v_y(x, y) = 0$ at each point of D.

Next, we consider $u(x, y)$ as a function of two real variables and approach it with some equipment from Calculus 3. Let P be a point in D and let P' be another point in D which lies on a line L which lies in D. Let $\mathbf{U}$ denote the unit vector along line L directed from P to P'. Let s denote the distance along L from point P. See Figure 2.30.

Theorem 2.24.A (continued 1)

Proof (continued).

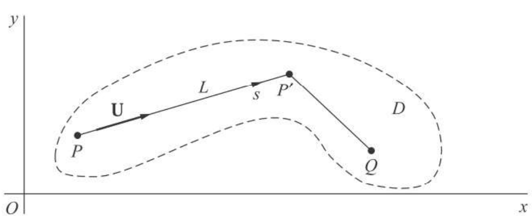


FIGURE 30

The directional derivative of $u(x, y)$ along line L is then $\frac{du}{ds} = \text{grad}(u) \cdot \mathbf{U}$ where $\text{grad}(u) = \nabla u = u_x(x, y)\mathbf{i} + u_y(x, y)\mathbf{j}$ (see Theorem 9 in my Calculus 3 (MATH 2110) notes on 14.5. Directional Derivatives and Gradient Vectors). Since $u_x(x, y) = u_y(x, y) = 0$ for all $(x, y) \in D$, then $\text{grad}(u) = \mathbf{0}$ at all points along L. So u is constant on L and the value of u at point P is the same as its value at P'.

Theorem 2.24.A (continued 2)

Theorem 2.24.A. If $f'(z) = 0$ everywhere in a domain D, then f must be constant throughout D.

Proof (continued). Since D is an open connected set, then any two points in D can be joined by a sequence of line segments in D (the is Theorem II.2.3 in Conway's *Functions of One Complex Variable I*; see my notes for Complex Analysis 1 on II.2. Connectedness). So if P and Q are any two points in D, then there is a sequence of line segments in D, say $\overline{PP_1}, \overline{P_1P_2}, \ldots, \overline{P_nQ}$, joining P to Q. As argued above, the value of u is the same at each of the points $P, P_1, P_2, \ldots, P_n, Q$ and so the value of u is the same at P and Q. Since P and Q are arbitrary points in D, then u is constant on D, say $u(x, y) = a$ for all $(x, y) \in D$.

Similarly, since $v_x(x, y) = v_y(x, y) = 0$ on D, then $v(x, y)$ is constant on D, say $v(x, y) = b$ for all $(x, y) \in D$. Therefore f is constant on D and $f(z) = a + ib$ for some $a + ib \in \mathbb{C}$. □