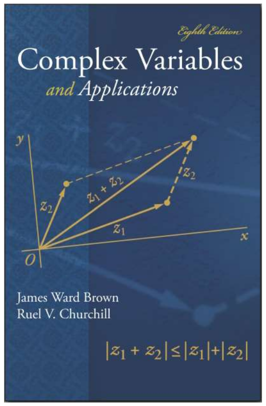
# Complex Variables

**Chapter 2. Analytic Functions**

Section 2.25. Examples—Proofs of Theorems








## Theorem 2.25.A

**Theorem 2.25.A.**
Suppose that a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a given domain $D$. Then $f$ is constant throughout $D$.

**Proof.** Write $\overline{f(z)} = U(x, y) + iV(x, y)$, so that

$$U(x, y) = u(x, y) \text{ and } V(x, y) = -v(x, y). \quad (1)$$

Since $f$ is analytic in $D$ then the Cauchy-Riemann equations are satisfied, by Theorem 2.21.A, and so

$$u_x(x, y) = v_y(x, y) \text{ and } u_y(x, y) = -v_x(x, y) \quad (2)$$

for all $(x, y)$ in $D$. Since $\overline{f}$ is analytic in $D$, then the Cauchy-Riemann equations also give $U_x(x, y) = V_y(x, y)$ and $U_y(x, y) = -V_x(x, y)$ for all $(x, y)$ in $D$. By equations (1), we therefore have throughout $D$ that

$$u_x(x, y) = -v_y(x, y) \text{ and } u_y(x, y) = -(-v_x(x, y)) \quad (4)$$





## Theorem 2.25.A (continued)

**Theorem 2.25.A.**
Suppose that a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a given domain $D$. Then $f$ is constant throughout $D$.

**Proof (continued).** Throughout $D$ we have:

$$u_x(x, y) = v_y(x, y) \text{ and } u_y(x, y) = -v_x(x, y) \quad (2)$$

$$u_x(x, y) = -v_y(x, y) \text{ and } u_y(x, y) = v_x(x, y) \quad (4)$$

Adding the respective sides of the first equations in (2) and (4) yields $u_x = 0$ on $D$. Subtracting the respective sides of the second equations in (2) and (4) yields $v_x = 0$ on $D$. So by Theorem 2.21.A, $f'(z) = u_x(x, y) + iv_x(x, y) = 0$ and so by Theorem 2.24.A, $f$ is constant on $D$. □





## Theorem 2.25.B

**Theorem 2.25.B.**
Suppose that a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a given domain $D$ and that $|f(z)|$ is constant throughout $D$. Then $f$ is constant throughout $D$.

**Proof.** Let $|f(z)| = c$ for all $z \in D$, where $c$ is a real nonnegative constant. If $c = 0$, the result follows. If $c \neq 0$, then $f(z)\overline{f(z)} = |f(z)|^2 = c^2$ and so $f(z) \neq 0$ in $D$. Hence $\overline{f(z)} = c^2/f(z)$ for all $z \in D$, and so $\overline{f(z)}$ is analytic in $D$ by Lemma 2.24.A(ii). So by Theorem 2.25.A, $f$ is constant throughout $D$. □