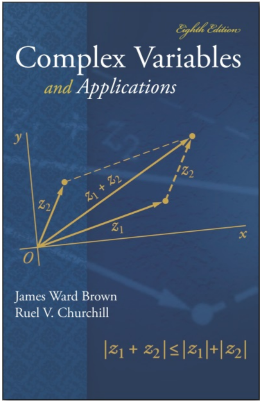
Complex Variables

Chapter 2. Analytic Functions

Section 2.25. Examples—Proofs of Theorems



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Theorem 2.25.A

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Suppose that a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a given domain D. Then f is constant throughout D.

Proof. Write $\overline{f(z)} = U(x, y) + iV(x, y)$, so that

$$U(x, y) = u(x, y) \text{ and } V(x, y) = -v(x, y). \quad (1)$$

Since f is analytic in D then the Cauchy-Riemann equations are satisfied, by Theorem 2.21.A, and so

$$u_x(x, y) = v_y(x, y) \text{ and } u_y(x, y) = -v_x(x, y) \quad (2)$$

for all (x, y) in D.

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for all (x, y) in D. Since $\overline{f}$ is analytic in D, then the Cauchy-Riemann equations also give $U_x(x, y) = V_y(x, y)$ and $U_y(x, y) = -V_x(x, y)$ for all (x, y) in D. By equations (1), we therefore have throughout D that

$$u_x(x, y) = -v_y(x, y) \text{ and } u_y(x, y) = -(-v_x(x, y)) \quad (4)$$

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Theorem 2.25.B.
Suppose that a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a given domain D and that $|f(z)|$ is constant throughout D. Then f is constant throughout D.

Proof. Let $|f(z)| = c$ for all $z \in D$, where c is a real nonnegative constant. If $c = 0$, the result follows. If $c \neq 0$, then $f(z)\overline{f(z)} = |f(z)|^2 = c^2$ and so $f(z) \neq 0$ in D. Hence $\overline{f(z)} = c^2/f(z)$ for all $z \in D$, and so $\overline{f(z)}$ is analytic in D by Lemma 2.24.A(ii).

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