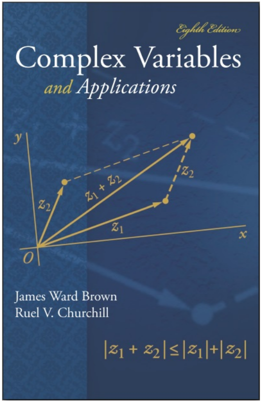
Complex Variables

Chapter 2. Analytic Functions

Section 2.26. Harmonic Functions—Proofs of Theorems



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Proof. In Corollary 4.52.A, we will see that if $f(z) = u(x, y) + iv(x, y)$ is analytic at a point then $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of all orders at the point. Since f is analytic in D then by the definition of "analytic" f is differentiable on D and so the Cauchy-Riemann equations are satisfied by Theorem 2.21.A. So $u_x = v_y$ and $u_y = -v_x$ on D.

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$$u_{xx} = v_{yx} \text{ and } u_{yx} = -v_{xx}. \qquad (3)$$

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By “The Mixed Derivative Theorem (Clairaut’s Theorem)” (see Theorem 2 of my Calculus 3 [MATH 2110] notes on 14.3. Partial Derivatives) if the first partials and the mixed second partials are continuous then the mixed second partials are equal. So $u_{xy} = u_{yx}$ and $v_{yx} = v_{xy}$.

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$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0.$$

So $u(x, y)$ and $v(x, y)$ are harmonic in D. □

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Theorem 2.26.2

Theorem 2.26.2. A function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if $v(x, y)$ is a harmonic conjugate of $u(x, y)$.

Proof. If v is a harmonic conjugate of u, then their first order partial derivatives satisfy the Cauchy-Riemann equations (by definition of harmonic conjugates) throughout D. So by Theorem 2.22.A, f is differentiable throughout D and so f is analytic on D.

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If f is analytic in D, then by Theorem 2.26.1 u and v are harmonic in D. By the definition of analytic, f is differentiable throughout D and so by Theorem 2.21.A, u and v satisfy the Cauchy-Riemann equations on D. So (by the definition of harmonic conjugates), v is a harmonic conjugate of v. □

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