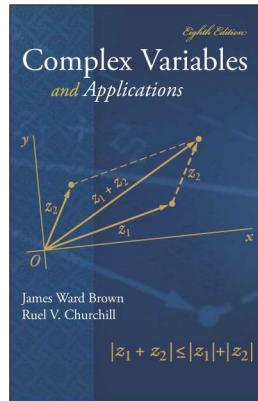


Complex Variables

Chapter 2. Analytic Functions

Section 2.27. Uniquely Determined Analytic Functions—Proofs of Theorems



Lemma 2.27.A

Lemma 2.27.A. Suppose that

- (a) a function f is analytic throughout a domain D ;
- (b) $f(z) = 0$ at each point z of a domain or line segment contained in D .

Then $f(z) = 0$ in D ; that is, $f(z)$ is identically equal to zero throughout D .

Proof. Let z_0 be any point of the domain or line segment on which $f(z) = 0$. Since D is a domain (an open connected set), then there is a polygonal line L consisting of a finite number of line segments joined end to end lying entirely in D that extends from z_0 to any other point P in D (see Theorem II.2.3 in my online notes for Complex Analysis 1 [MATH 5510] on II.2. Connectedness; this was also used in the proof of Theorem 2.24.A). Let d be the shortest distance from points on L to the boundary of D (that is, d is the distance from the set of points comprising the boundary of D and the set of points comprising the polygonal line; there is a “shortest” distance since the polygonal line is a compact set; . . .

Lemma 2.27.A (continued)

Proof (continued). . . . see my online class notes for Complex Analysis 1 [MATH 5510] on II.5. Continuity), unless D is the entire complex plane; in that case, d may be any positive real number. We then form a finite sequence of points $z_0, z_1, z_2, \dots, z_{n-1}, z_n = P$ where each point lies on polygonal line L and $|z_k - z_{k-1}| < d$ for $k = 1, 2, \dots, n$ (such finite collection of points exists because L is a compact set). See Figure 2.33.

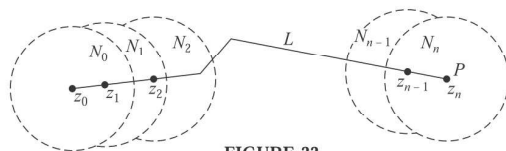


FIGURE 33

Construct a sequence of neighborhoods $N_0, N_1, \dots, N_{n-1}, N_n$ where N_k is centered at z_k and has radius d (so $N_k = \{z \in \mathbb{C} \mid |z - z_k| < d\}$). Then all these neighborhoods are contained in D and that $z_k \in N_{k-1}$ for $k = 1, 2, \dots, n$.

Lemma 2.27.A (continued)

Proof (continued). We now borrow a result from Section 6.75:

Theorem 6.75.3. Given a function f and a point z_0 , suppose that

- (a) f is analytic throughout a neighborhood N_0 of z_0 ;
- (b) $f(z) = 0$ at each point z of a domain D or line segment L containing z_0 .

Then $f(z) \equiv 0$ in N_0 ; that is, $f(z)$ is identically zero throughout N_0 .

Since z_0 is a point of the domain or line segment on which $f(z) = 0$, so N_0 contains a domain or line segment on which $f(z) = 0$ (namely, the intersection of N_0 with the domain or line segment). By Theorem 6.75.3, $f(z) \equiv 0$ on N_0 . Next, $N_0 \cap N_1$ is a domain in N_1 on which $f(z) = 0$, so by Theorem 6.75.3 $f(z) \equiv 0$ on N_1 . Similarly, $f(z) \equiv 0$ on $N_2 \cup N_3 \cup \dots \cup N_n$. Therefore, $f(z_n) = f(P) = 0$. Since P was an arbitrary point of D , then $f(z) \equiv 0$ throughout D . □

Theorem 2.27.A

Theorem 2.27.A. A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D .

Proof. Suppose two functions f and g are analytic in domain D and that $f(z) = g(z)$ for all z in some domain or line segment contained in D . Then $h(z) = f(z) - g(z)$ is analytic in F and $h(z) = 0$ throughout some domain or line segment contained in D . By Lemma 2.27.A, $h(z) \equiv 0$ throughout D , and so $f(z) = g(z)$ for all $z \in D$. \square