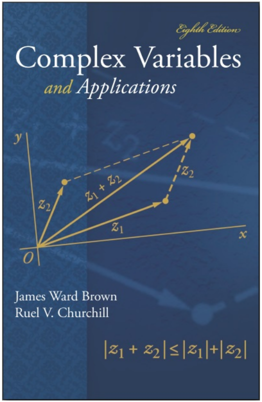
Complex Variables

Chapter 2. Analytic Functions

Section 2.27. Uniquely Determined Analytic Functions—Proofs of Theorems



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(a) a function f is analytic throughout a domain D;

(b) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Then $f(z) = 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

Proof. Let z_0 be any point of the domain or line segment on which $f(z) = 0$. Since D is a domain (an open connected set), then there is a polygonal line L consisting of a finite number of line segments joined end to end lying entirely in D that extends from z_0 to any other point P in D (see Theorem II.2.3 in my online notes for Complex Analysis 1 [MATH 5510] on II.2. Connectedness; this was also used in the proof of Theorem 2.24.A).

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Lemma 2.27.A (continued)

Proof (continued). ...see my online class notes for Complex Analysis 1 [MATH 5510] on II.5. Continuity), unless D is the entire complex plane; in that case, d may be any positive real number. We then form a finite sequence of points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n = P$ where each point lies on polygonal line L and $|z_k - z_{k-1}| < d$ for $k = 1, 2, \ldots, n$ (such finite collection of points exists because L is a compact set). See Figure 2.33.

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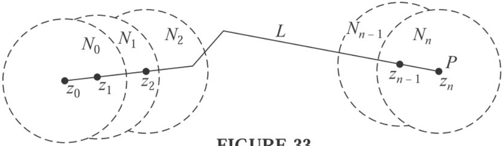

FIGURE 33

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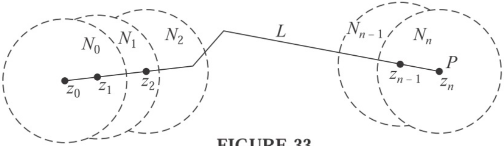

FIGURE 33

Construct a sequence of neighborhoods $N_0, N_1, \ldots, N_{n-1}, N_n$ where N_k is centered at z_k and has radius d (so $N_k = \{z \in \mathbb{C} \mid |z - z_k| < d\}$). Then all these neighborhoods are contained in D and that $z_k \in N_{k-1}$ for $k = 1, 2, \ldots, n$.

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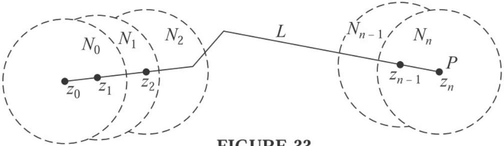

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Proof (continued). We now borrow a result from Section 6.75:

Theorem 6.75.3. Given a function f and a point z_0, suppose that

(a) f is analytic throughout a neighborhood N_0 of z_0;

(b) $f(z) = 0$ at each point z of a domain D or line segment L containing z_0.

Then $f(z) \equiv 0$ in N_0; that is, $f(z)$ is identically zero throughout N_0.

Since z_0 is a point of the domain or line segment on which $f(z) = 0$, so N_0 contains a domain or line segment on which $f(z) = 0$ (namely, the intersection of N_0 with the domain or line segment). By Theorem 6.75.3, $f(z) \equiv 0$ on N_0. Next, $N_0 \cap N_1$ is a domain in N_1 on which $f(z) = 0$, so by Theorem 6.75.3 $f(z) \equiv 0$ on N_1. Similarly, $f(z) \equiv 0$ on $N_2 \cup N_3 \cup \cdots \cup N_n$. Therefore, $f(z_n) = f(P) = 0$. Since P was an arbitrary point of D, then $f(z) \equiv 0$ throughout D. □

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Theorem 2.27.A. A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.

Proof. Suppose two functions f and g are analytic in domain D and that $f(z) = g(z)$ for all z in some domain or line segment contained in D. Then $h(z) = f(z) - g(z)$ is analytic in F and $h(z) = 0$ throughout some domain of line segment contained in D. By Lemma 2.27.A, $h(z) \equiv 0$ throughout D, and so $f(z) = g(z)$ for all $z \in D$. □

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