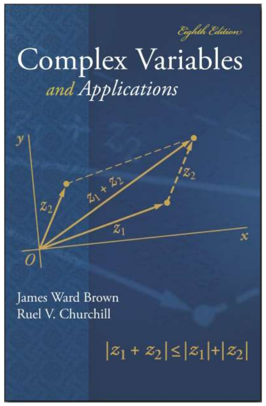
Complex Variables

Chapter 2. Analytic Functions

Section 2.28. Reflection Principle—Proofs of Theorems



Theorem 2.28.A

Theorem 2.28.A. Reflection Principle. Suppose that a function f is analytic in some domain D which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to that axis. Then $\overline{f(z)} = f(\overline{z})$ for each point z in the domain if and only if $f(x)$ is real for each point x on that segment.

Proof. Suppose $f(x)$ is real for each real x in the segment. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $F(z) = \overline{f}(z) = \overline{f(\overline{z})} = U(x,y) + iV(x,y)$. So

$$\overline{f(\overline{z})} = U(x,y) + iV(x,y) = u(x,-y) - iv(x,-y).$$

Since f is analytic on D then $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations by Theorem 2.21.A and so $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$. Since $U(x,y) = -u(x,-y)$ then $U_x(x,y) = u_x(x,-y)$ and $U_y(x,y) = -u_y(x,-y)$. Since $V(x,y) = -v(x,-y)$ then $V_x(x,y) = -v_x(x,-y)$ and $V_y(x,y) = v_y(x,-y)$.

Theorem 2.28.A (continued 1)

Proof (continued). Next, $u_x(x,y) = v_y(x,y)$ implies $u_x(x,-y) = v_y(x,-y)$ or that $U_x(x,y) = V_y(x,y)$. Similarly $u_x(x,y) = -v_x(x,y)$ implies that $u_y(x,-y) = -v_x(x,-y)$ or that $-U_y(x,y) = V_x(x,y)$. So $U(x,y)$ and $V(x,y)$ satisfy the Cauchy-Riemann equations and have continuous partial derivatives (since u and v have continuous partial derivatives) throughout D, therefore by Theorem 2.22.A $F(z) = \overline{f(\overline{z})}$ is analytic in D.

Since $f(x) = f(x+i0)$ is real on the segment of the real axis lying in D, then $v(x,0) = 0$ for all x on the segment. Therefore, for such x,

$$F(x) = U(x,0) + iV(x,0) = u(x,0) - iv(x,0) = u(x,0) = f(x).$$

So for x on the segment of the real axis, $F(z) = f(z)$. By Theorem 2.27.A we have that $F(z) = f(z)$ for all $z \in D$. Hence $\overline{f(\overline{z})} = f(z)$ for all $z \in D$. Taking conjugates of both sides, we have $f(\overline{z}) = \overline{f(z)}$ for all $z \in D$, as claimed.

Theorem 2.28.A (continued 2)

Theorem 2.28.A. Reflection Principle. Suppose that a function f is analytic in some domain D which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to that axis. Then $\overline{f(z)} = f(\overline{z})$ for each point z in the domain if and only if $f(x)$ is real for each point x is that segment.

Proof (continued). Conversely, suppose $f(\overline{z}) = \overline{f(z)}$ for all $z \in D$. With the notation above, we have

$$f(\overline{z}) = u(x,-y) + iv(x,-y) = u(x,y) - iv(x,y) = \overline{f(z)}.$$

With $z = x + iy = x + i0$ real and in D we then have $u(x,0) + iv(x,0) = u(x,0) - iv(x,0)$ and so $v(x,0) = -v(x,0)$ or $v(x,0) = 0$. That is, for any real $z = x \in D$ we have that $f(z) = f(x) = u(x,0)$ is real, as claimed. □