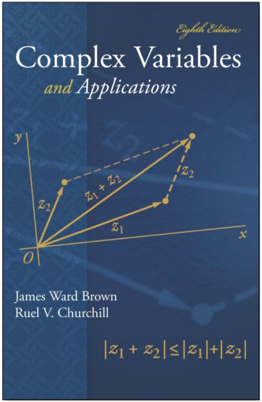
Complex Variables

Chapter 2. Analytic Functions

Section 2.28. Reflection Principle—Proofs of Theorems



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Proof. Suppose $f(x)$ is real for each real x in the segment. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and $F(z) = \overline{f}(z) = \overline{f(\overline{z})} = U(x, y) + iV(x, y)$. So

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$$\overline{f(\overline{z})} = U(x,y) + iV(x,y) = u(x,-y) - iv(x,-y).$$

Since f is analytic on D then $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations by Theorem 2.21.A and so $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$. Since $U(x,y) = -u(x,-y)$ then $U_x(x,y) = u_x(x,-y)$ and $U_y(x,y) = -u_y(x,-y)$.

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Proof (continued). Next, $u_x(x, y) = v_y(x, y)$ implies $u_x(x, -y) = v_y(x, -y)$ or that $U_x(x, y) = V_y(x, y)$. Similarly $u_x(x, y) = -v_x(x, y)$ implies that $u_y(x, -y) = -v_x(x, -y)$ or that $-U_y(x, y) = V_x(x, y)$. So $U(x, y)$ and $V(x, y)$ satisfy the Cauchy-Riemann equations and have continuous partial derivatives (since u and v have continuous partial derivatives) throughout D, therefore by Theorem 2.22.A $F(z) = \overline{f(\overline{z})}$ is analytic in D.

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Since $f(x) = f(x + i0)$ is real on the segment of the real axis lying in D, then $v(x, 0) = 0$ for all x on the segment. Therefore, for such x,

$$F(x) = U(x, 0) + iV(x, 0) = u(x, 0) - iv(x, 0) = u(x, 0) = f(x).$$

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