Complex Variables

Chapter 3. Elementary Functions

Section 3.29. The Exponential Function—Proofs of Theorems

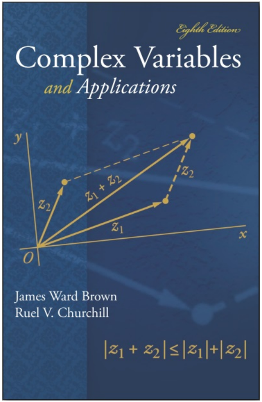

Table of contents

Lemma 3.29

Lemma 3.29.A. For all $z_1, z_2 \in \mathbb{C}$ we have $e^{z_1} e^{z_2} = e^{z_1+z_2}$.

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, as in the proof of Theorem 1.7.1,

$$\begin{aligned} e^{z_1} e^{z_2} &= (e^{x_1}(\cos y_1 + i \sin y_1))(e^{x_2}(\cos y_2 + i \sin y_2)) \\ &= e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) \\ &\quad + e^{x_1+x_2} i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2) \\ &= e^{x_1+x_2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)) \text{ since} \\ &\quad \cos(y_1 \pm y_2) = \cos y_1 \cos y_2 \mp \sin y_1 \sin y_2 \\ &\quad \text{ and } \sin(y_1 \pm y_2) = \sin y_1 \cos y_2 \pm \cos y_1 \sin y_2 \\ &= e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}. \end{aligned}$$

□

Lemma 3.29

Lemma 3.29.A. For all $z_1, z_2 \in \mathbb{C}$ we have $e^{z_1} e^{z_2} = e^{z_1+z_2}$.

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, as in the proof of Theorem 1.7.1,

$$\begin{aligned}
e^{z_1} e^{z_2} &= (e^{x_1}(\cos y_1 + i \sin y_1))(e^{x_2}(\cos y_2 + i \sin y_2)) \\
&= e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) \\
&\quad + e^{x_1+x_2} i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2) \\
&= e^{x_1+x_2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)) \text{ since} \\
&\quad \cos(y_1 \pm y_2) = \cos y_1 \cos y_2 \mp \sin y_1 \sin y_2 \\
&\quad \text{ and } \sin(y_1 \pm y_2) = \sin y_1 \cos y_2 \pm \cos y_1 \sin y_2 \\
&= e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}.
\end{aligned}$$

□

Corollary 3.29.A

Corollary 3.29.A. For all $z_1, z_2 \in \mathbb{C}$ we have $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$. Also, $1/e^z = (e^z)^{-1} = e^{-z}$.

Proof. By Lemma 3.29.A, we have $e^{z_1-z_2}e^{z_2} = e^{(z_1-z_2)+z_2} = e^{z_1}$ and rearranging this gives $e^{z_1}/e^{z_2} = e^{z_1-z_2}$, as claimed. Since $e^0 = 1$, this implies (with $z_1 = 0$ and $z_2 = z$) that $1/e^z = e^0/e^z = e^{0-z} = e^{-z}$ (notice $1/e^z = (e^z)^{-1}$ are just alternative notations for multiplicative inverses, whereas e^{-z} denotes the exponential function evaluated at $-z$; this result shows the relationship between the multiplicative inverse and the value of e^{-z}). □

Corollary 3.29.A

Corollary 3.29.A. For all $z_1, z_2 \in \mathbb{C}$ we have $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$. Also, $1/e^z = (e^z)^{-1} = e^{-z}$.

Proof. By Lemma 3.29.A, we have $e^{z_1 - z_2} e^{z_2} = e^{(z_1 - z_2) + z_2} = e^{z_1}$ and rearranging this gives $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$, as claimed. Since $e^0 = 1$, this implies (with $z_1 = 0$ and $z_2 = z$) that $1/e^z = e^0/e^z = e^{0-z} = e^{-z}$ (notice $1/e^z = (e^z)^{-1}$ are just alternative notations for multiplicative inverses, whereas e^{-z} denotes the exponential function evaluated at $-z$; this result shows the relationship between the multiplicative inverse and the value of e^{-z}). □