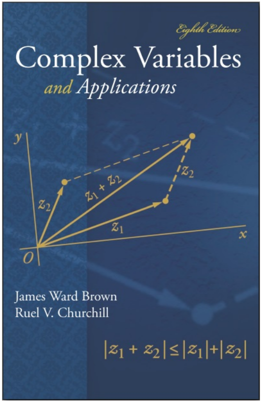
Complex Variables

Chapter 3. Elementary Functions

Section 3.32. Some Identities Involving Logarithms—Proofs of Theorems



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Lemma 3.32.A

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$$\log(z_1 z_2) = \log z_1 + \log z_2.$$

Proof. Since we have by definition, $\log z = \ln |z| + i\arg(z)$, and by Lemma 1.8.1, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, then

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Lemma 3.32.C

Lemma 3.32.C. For any nonzero $z \in \mathbb{C}$, for all $n \in \mathbb{Z}$ we have $z^n = e^{n \log z}$.

Proof. First, for $z = re^{i\theta} = |z|e^{i\arg z}$, notice that $e^{\log z} = e^{\ln|z| + i\arg z} = e^{\ln|z|}e^{i\arg z} = |z|e^{i\arg z} = z$. So for $n \geq 0$ we have

$$\begin{aligned} z^n &= (e^{\log z})^n = \underbrace{(e^{\log z})(e^{\log z})\cdots(e^{\log z})}_{n \text{ times}} \\ &= e^{n \log z} \text{ by Lemma 3.29.A.} \end{aligned}$$

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For $n < 0$ we have

$$\begin{aligned} z^n &= (e^{\log z})^n = (e^{-\log z})^{-n} = \underbrace{(e^{-\log z})(e^{-\log z})\cdots(e^{-\log z})}_{-n \text{ times}} \\ &= e^{-(-n)\log z} \text{ by Lemma 3.29.A} \\ &= e^{n \log z}. \end{aligned}$$

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Lemma 3.32.D

Lemma 3.32.D. For any nonzero $z \in \mathbb{C}$, we have that for $n = 1, 2, 3, \ldots$

$$\exp\left(\frac{1}{n}\log z\right)$$

is a set consisting of n distinct elements each of which is an nth root of z (that is, when raised to the nth power gives z).

Proof. Let $z = r\exp(i\Theta) = |z|\exp(i\Theta)$ where Θ is the principal value of $\arg(z)$ (that is, $\Theta \in \arg(z)$ and $-\pi < \Theta \leq \pi$). Then

$$\begin{aligned}\exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}(\ln|z| + i(\Theta + 2k\pi))\right) \text{ where } k \in \mathbb{Z} \\ &= \exp\left(\frac{1}{n}\ln|z| + i\frac{\Theta + 2k\pi}{n}\right) \\ &= \exp\left(\frac{1}{n}\ln|z|\right)\exp\left(i\frac{\Theta + 2k\pi}{n}\right) \text{ by Lemma 3.29.A.} \quad (7)\end{aligned}$$

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Proof (continued). Now $\exp\left(i\left(\Theta/n + 2k\pi/n\right)\right) = \cos(\Theta/n + 2k\pi/n) + i\sin(\Theta/n + 2k\pi/n)$ and this results in n distinct values as k ranges over the distinct values modulo n (say, $k = 0, 1, \ldots, n-1$). For each value given in (7), we have

$$\begin{aligned} \left[\exp\left(\frac{1}{n}\log z\right)\right]^n &= \left[\exp\left(\frac{1}{n}\ln|z|\right)\exp\left(i\frac{\Theta + 2k\pi}{n}\right)\right]^n \\ &= \left[\exp\left(\frac{1}{n}\ln|z|\right)\right]^n \left[\exp\left(i\frac{\Theta + 2k\pi}{n}\right)\right]^n \\ &= \exp(\ln|z|)\exp(i(\Theta + 2k\pi)) \text{ by Lemma 3.29.A} \\ &= |z|e^{i(\Theta+2k\pi)} = |z|e^{i\Theta} = z. \end{aligned}$$

So the result follows, which we denote as $z^{1/n} = \exp\left(\dfrac{1}{n}\log z\right)$. □

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&= \exp(\ln|z|)\exp(i(\Theta + 2k\pi)) \text{ by Lemma 3.29.A} \\
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