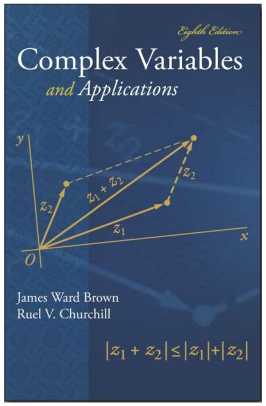
Complex Variables

Chapter 3. Elementary Functions

Section 3.33. Complex Exponents—Proofs of Theorems



Theorem 3.33.A

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Proof. Let $\log z$ represent some branch of the logarithm so that $\log z = \ln|z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$ for some given α. Then

$$\begin{aligned}\frac{d}{dz}[z^c] &= \frac{d}{dz}[\exp(c\log z)] \\ &= c\frac{1}{z}[\exp(c\log z)] \text{ since } \frac{d}{dz}[\log z] = 1/z \text{ as shown in Note 3.31.A} \\ &\quad\ \text{and by the Chain Rule (Theorem 2.20.C)} \\ &= c\frac{\exp(c\log z)}{\exp(\log z)} \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A} \\ &= \cdots\end{aligned}$$

Theorem 3.33.A (continued)

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Proof (continued). $\cdots$

$$\begin{aligned}\frac{d}{dz}[z^c] &= c\frac{\exp(c\log z)}{\exp(\log z)} \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A} \\ &= c\exp(c\log z)\exp(-\log z) \text{ by Corollary 3.29.A} \\ &= c\exp(c\log z - \log z) = c\exp((c-1)\log z) \text{ by Lemma 3.29.A} \\ &= cz^{c-1} \text{ for the branch of } z^{c-1} \text{ based on} \\ &\quad\ \text{the branch of the logarithm } \log z.\end{aligned}$$

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