Complex Variables

Chapter 3. Elementary Functions

Section 3.33. Complex Exponents—Proofs of Theorems

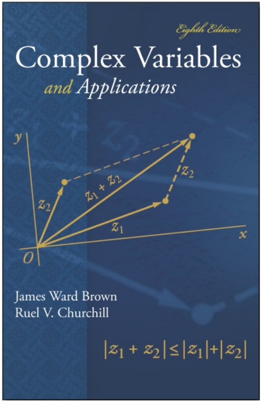

Table of contents

Theorem 3.33.A

Theorem 3.33.A. For any branch of z^c, we have $\frac{d}{dz}[z^c] = cz^{c-1}$ where the branch of z^{c-1} is based on the same branch of the logarithm on which z^c is based.

Proof. Let $\log z$ represent some branch of the logarithm so that $\log z = \ln |z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$ for some given α. Then

$$\begin{aligned} \frac{d}{dz}[z^c] &= \frac{d}{dz}[\exp(c \log z)] \\ &= c\frac{1}{z}[\exp(c \log z)] \text{ since } \frac{d}{dz}[\log z] = 1/z \text{ as shown in Note 3.31.A} \\ &\quad \text{and by the Chain Rule (Theorem 2.20.C)} \\ &= c\frac{\exp(c \log z)}{\exp(\log z)} \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A} \\ &= \cdots \end{aligned}$$

Theorem 3.33.A

Theorem 3.33.A. For any branch of z^c, we have $\frac{d}{dz}[z^c] = cz^{c-1}$ where the branch of z^{c-1} is based on the same branch of the logarithm on which z^c is based.

Proof. Let $\log z$ represent some branch of the logarithm so that $\log z = \ln|z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$ for some given α. Then

$$\begin{aligned} \frac{d}{dz}[z^c] &= \frac{d}{dz}[\exp(c\log z)] \\ &= c\frac{1}{z}[\exp(c\log z)] \text{ since } \frac{d}{dz}[\log z] = 1/z \text{ as shown in Note 3.31.A} \\ &\quad \text{ and by the Chain Rule (Theorem 2.20.C)} \\ &= c\frac{\exp(c\log z)}{\exp(\log z)} \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A} \\ &= \cdots \end{aligned}$$

Theorem 3.33.A (continued)

Theorem 3.33.A. For any branch of z^c, we have $\frac{d}{dz}[z^c] = cz^{c-1}$ where the branch of z^{c-1} is based on the same branch of the logarithm on which z^c is based.

Proof (continued). $\cdots$

$$\begin{aligned}
\frac{d}{dz}[z^c] &= c\frac{\exp(c\log z)}{\exp(\log z)} \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A} \\
&= c\exp(c\log z)\exp(-\log z) \text{ by Corollary 3.29.A} \\
&= c\exp(c\log z - \log z) = c\exp((c-1)\log z) \text{ by Lemma 3.29.A} \\
&= cz^{c-1} \text{ for the branch of } z^{c-1} \text{ based on} \\
&\quad \text{ the branch of the logarithm } \log z.
\end{aligned}$$

□