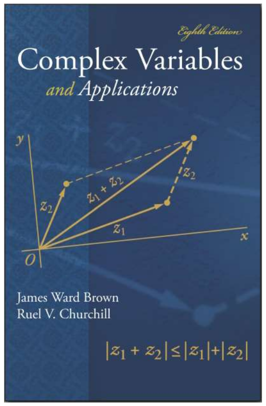
Complex Variables

Chapter 3. Elementary Functions

Section 3.34. Trigonometric Functions—Proofs of Theorems



Lemma 3.34.A

Lemma 3.34.A. The real and imaginary parts of $\cos z$ and $\sin z$ can be expressed in terms of $\sin x$, $\cos x$, $\sinh y$, and $\cosh y$, where $z = x + iy$, as:

$$\sin z = \sin x \cosh y + i \cos x \sinh y \text{ and } \cos z = \cos x \cosh y - i \sin x \sinh y.$$

Proof. Let $z_1 = x$ and $z_2 = iy$. Then by the summation equations we have

$$\begin{aligned} \sin z &= \sin(x + iy) = \sin(z_1 + z_2) \\ &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + \cos x (i \sinh y) \text{ since } \cos iy = \cosh y \\ &\quad \text{and } \sin iy = i \sinh y \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

Lemma 3.34.A (continued)

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Proof (continued). We know that the derivative of $\sin z$ is $\cos z$ and for $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ we have $f'(z) = f'(x + iy) = u_x(x, y) + iv_x(x, y)$ by Theorem 2.21.A. So

$$\cos z = \frac{d}{dz}[\sin z] = \frac{\partial}{\partial x}[\sin x \cosh y] + i \frac{\partial}{\partial x}[\cos x \sinh y]$$

$$= \cos x \cosh y - i \sin x \sinh y.$$

□

Lemma 3.34.B

Lemma 3.34.B. The only zeros of $\sin z$ are the real numbers $z = n\pi$ where $n \in \mathbb{Z}$. The only zeros of $\cos z$ are the real numbers $z = \pi/2 + n\pi$ where $n \in \mathbb{Z}$.

Proof. Since $\cos z$ and $\sin z$ equal $\cos x$ and $\sin x$ (respectively) on the real axis, then all the zeros of the real function are also zeros of the corresponding complex function. Now suppose $z = x + iy$ is a zero of $\sin z$. Then it must be that $|\sin z|^2 = 0$ and we have from Note 3.34.C that $|\sin z|^2 = \sin^2 x + \sinh^2 y$. So we must have $\sin x = \sinh y = 0$. But the only value of y for which $\sinh y = (e^y - e^{-y})/2 = 0$ is $y = 0$. So the zeros of $\sin z$ are those values of $x \in \mathbb{R}$ for which $\sin x = 0$; namely, the zeros of $\sin z$ are all $z = n\pi$ where $n \in \mathbb{Z}$.

Lemma 3.34.B (continued)

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Proof (continued). By the summation formula for sine,

$$-\sin(z - \pi/2) = -(\sin z \cos(-\pi/2) + \cos z \sin(-\pi/2))$$

$$= -(\sin z)0 - \cos z(-1) = \cos z.$$

So $\cos z = 0$ if and only if $\sin(z - \pi/2) = 0$; that is, if and only if $z - \pi/2 = n\pi$ where $n \in \mathbb{Z}$. So the zeros of $\cos z$ is $z = \pi/2 + n\pi$ where $n \in \mathbb{Z}$. □