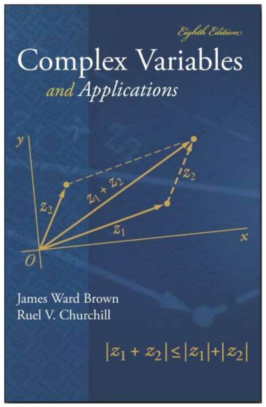
Complex Variables

Chapter 4. Integrals

Section 4.43. Upper Bounds for Moduli of Contour Integrals—Proofs of Theorems



Lemma 4.43.A

Lemma 4.43.A. If $w(t)$ is a piecewise continuous complex valued function defined on an interval $a \le t \le b$, then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt.$$

Proof. If $\int_a^b w(t)\,dt = 0$ then the result trivially holds so, without loss of generality, say $\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \neq 0$. Then $r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt \in \mathbb{R}$; that is

$$r_0 = \text{Re}\left(\int_a^b e^{-\theta_0} w(t)\,dt\right) = \int_a^b \text{Re}(e^{-i\theta_0} w(t))\,dt.$$

But $\text{Re}(e^{-i\theta_0} w(t)) \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}||w(t)| = |w(t)|$, so that $r_0 = \int_a^b \text{Re}(e^{-i\theta_0} w(t))\,dt \le \int_a^b |w(t)|\,dt$, or

$$r_0 = \left|\int_a^b e^{-i\theta_0} w(t)dt\right| = \left|e^{-i\theta_0}\int_a^b w(t)dt\right| = \left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|dt.$$

□

Theorem 4.43.A

Theorem 4.43.A. Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defined, then $\left|\int_C f(z)\,dz\right| \le ML$.

Proof. Let $C = \{z(t) \mid t \in [a,b]\}$. By Lemma 4.43.A

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t))z'(t)|\,dt.$$

Since $|f(z)| \le M$ for $z \in C$ then $|f(z(t))| \le M$ for $t \in [a,b]$ and so $\int_a^b |f(z(t))z'(t)|\,dt \le M\int_a^b |z'(t)|\,dt$ and hence

$$\left|\int_C f(z)\,dz\right| \le M\int_a^b |z'(t)|\,dt = ML,$$

since $L = \int_a^b |z'(t)|\,dt$ by definition (see Section 39). □