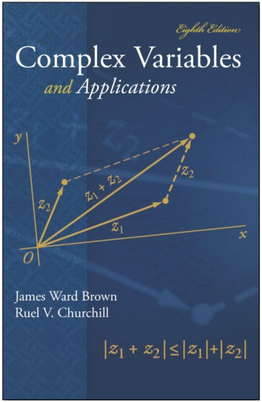
Complex Variables

Chapter 4. Integrals

Section 4.47. Proof of the Theorem—Proofs of Theorems



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Lemma 4.47.1

Lemma 4.47.1. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any $\varepsilon > 0$, the region R can be covered with a finite number of squares and partial squares indexed by $j = 1, 2, \ldots, n$ such that in each one there is a fixed point z_j for which the inequality

$$\left| \frac{f(z) - f(z_j)}{z - z_i} - f'(z_j) \right| < \varepsilon$$

is satisfied by all points other than z_j in that square or partial square.

Proof. Let $\varepsilon > 0$. ASSUME that R has been covered with a finite number of squares and partial squares but that one of the squares or partial squares violates the claim of the lemma; that is, some square or partial square cannot be subdivided enough so that the lemma is satisfied. Let σ_0 denote this subregion if it is a square, or let σ_0 denote the complete square which contains the partial square (so that in this second case σ_0 contains points in R and points not in R).

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Lemma 4.47.1 (continued 1)

Proof. Next, subdivide σ_0 into four squares (each of which has a side of length $1/2$ the length of a side of σ_0). At least one of the four smaller squares must contain points of R but no point z_j which satisfies the lemma. Denote this square as σ_1. Then inductively construct nested sets $\sigma_0, \sigma_1, \sigma_2, \ldots$ where each σ_k violated the lemma. By Exercise 4.49.9 (or Exercise 4.53.9 in the 9th edition of the book) there is some point z_0 common to all σ_k. Also each of these squares contains points of R other than z_0 (details could be added here). Now every δ neighborhood $|z - z_0| < \delta$ of z_0 contains the squares σ_k provided the diagonal of the σ_k has length less than δ.

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Lemma 4.47.1 (continued 2)

Proof. So there exists $\delta > 0$ such that for all z in the deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 we have

$$\left| \frac{f(z) - f(z_j)}{z - z_i} - f'(z_j) \right| < \varepsilon.$$

But the neighborhood $|z - z_0| < \delta$ of z_0 contains a square σ_K (when K is large enough that the length of a diagonal of that square is less than δ; see Figure 56).

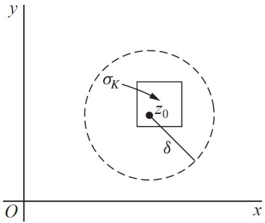

FIGURE 56

Lemma 4.47.1 (continued 3)

Lemma 4.47.1. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any $\varepsilon > 0$, the region R can be covered with a finite number of squares and partial squares indexed by $j = 1, 2, \ldots, n$ such that in each one there is a fixed point z_j for which the inequality $|(f(z) - f(z_j))(z - z_i) - f'(z_j)| < \varepsilon$ is satisfied by all points other than z_j in that square or partial square.

Proof (continued). But then z_0 serves as the point z_j in the claim of the lemma on the square σ_K (or the partial square $\sigma_K \cap R$). This CONTRADICTS the fact that σ_K was constructed in such a way that the lemma is not satisfied on square σ_K (or partial square $\sigma_K \cap R$). This contradiction shows that the assumption that some square σ_0 cannot be sufficiently subdivided enough so that the lemma is satisfied is false. So in the original covering of R by squares, each square can be subdivided enough to satisfy the lemma. This shows that the region R can be covered as required. □

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Theorem 4.47.A

Lemma 4.47.A. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any covering of R by squares and partial squares as described in Lemma 4.47.1, put positive orientations on each of the boundaries of the squares and partial squares (see Figure 57) and denote the resulting positively oriented contours as $C_1, C_2, \ldots, C_n$. On the jth square or partial square, define

$$\delta_j(z) = \begin{cases} (f(z) - f(z_j))/(z - z_j) - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j. \end{cases}$$

Then

$$\left| \int_C f(z)\,dz \right| \leq \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)\,dz \right|.$$

Theorem 4.47.A (continued 1)

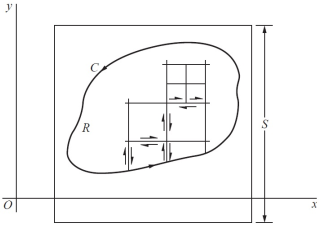


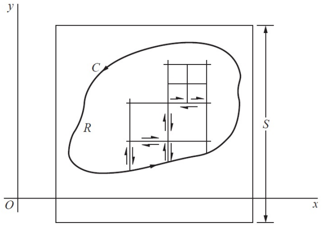
FIGURE 57

Proof. Given $\varepsilon > 0$, cover R by squares and partial squares as described in Lemma 4.47.1. Label the squares $1, 2, \ldots, n$ and on the jth square or partial square define

$$\delta_j(z) = \begin{cases} (f(z) - f(z_j))/(z - z_j) - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j. \end{cases}$$

Now $\delta_j(z)$ is continuous throughout the subregion since $f(z)$ is continuous there, and so $\lim_{z \to z_j} \delta_j(z) = f'(z_j) = f'(z_j) = 0$.

Theorem 4.47.A (continued 1)



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Theorem 4.47.A (continued 2)

Proof (continued). For $z \in C_j$, by the definition of $\delta_j(z)$, we have

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z)$$

and so

$$\int_{C_j} f(z)\,dz = (f(z_j) - z_j f'(z_j)) \int_{C_j} dz + f'(z_j) \int_{C_j} z\,dz + \int_{C_j} (z - z_j)\delta_j(z)\,dz.$$

But since C_j is closed then $\int_{C_j} dz = \int_{C_j} z\,dz = 0$ by Theorem 4.44.A(c), so we have

$$\int_{C_j} f(z)\,dz = \int_{C_j} (z - z_j)\delta_j(z)\,dz \text{ for } j = 1, 2, \ldots, n. \qquad (*)$$

Summing over j we have $\displaystyle\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$ since the two integrals along the common boundary of every pair of adjacent subregions cancel each other (see Figure 57), so only the integrals along the arcs that are parts of C remain in the sum.

Theorem 4.47.A (continued 2)

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Theorem 4.47.A (continued 3)

Proof (continued). By $(*)$ we therefore have

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)\,dz$$

and so by the Triangle Inequality,

$$\int_C f(z)\,dz \le \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)\,dz \right|,$$

as claimed. □

Proof of the Cauchy-Goursat Theorem

Proof. Let $\varepsilon > 0$ and let region R consist of the points interior to (positively oriented) simple closed contour C together with the points on C itself. Let there be a covering of R with squares and partial squares, as given by Lemma 4.47.1. Number the squares and partial squares with $1, 2, \ldots, n$. Let C_j be as described in Lemma 4.47.A and let s_j be the length of a side of square j. Then for z and z_j in the jth square (or partial square) we have $|z - z_j| \le \sqrt{2}s_j$. By the choice of the squares and definition of $\delta_j(z)$ in Lemma 4.47.A, we have $|\delta_j(z)| < \varepsilon$. So for $z \in C_j$,

$$|(z - z_j)\delta_j(z)| = |z - z_j||\delta_j(z)| < \sqrt{2}s_j\varepsilon.$$

Since the length of path C_j is $4s_j$ if C_j is the boundary of a square and in this case, by Theorem 4.43.A,

$$\left|\int_{C_j} (z - z_j)\delta_j(z)\, dz\right| < \sqrt{2}s_j\varepsilon 4s_j = 4\sqrt{2}A_j\varepsilon$$

where $A_j = s_j^2$ is the area of the jth square.

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Proof of the Cauchy-Goursat Theorem (continued 1)

Proof (continued). If C_j is the boundary of a partial square, its length does not exceed $4s_j + L_j$ where L_j is the length of the part of X_j which is also a part of C. Again by Theorem 4.43.A, we have

$$\left| \int_{C_1} (z - z_j)\delta_j(z)\, dz \right| < \sqrt{2} s_j \varepsilon (4s_j + L_j) < 4\sqrt{2} A_j \varepsilon + \sqrt{2} S L_k \varepsilon$$

where S is the length of a side of some square that encloses the entire contour C as well as all of the squares in covering R (see Figure 57). So the sum of all of the areas A_j does not exceed S^2.

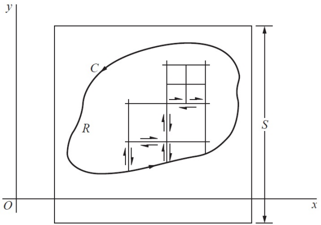

FIGURE 57

Proof of the Cauchy-Goursat Theorem (continued 2)

Proof (continued). If L denotes the length of contour C, then we have

$$\begin{aligned}\left|\int_C f(z)\,dz\right| &\le \sum_{j=1}^{n}\left|\int_{C_j}(z-z_j)\delta_j(z)\,dz\right| \text{ by Lemma 4.47.A}\\ &< 4\sqrt{2}\varepsilon{\sum}' A_j + \sqrt{2}\varepsilon\left(4{\sum}'' A_j + S{\sum}'' L_j\right)\\ &\quad \text{where } {\sum}' \text{ denotes summation over indices}\\ &\quad \text{involving squares and } {\sum}'' \text{ denotes summation}\\ &\quad \text{over indices involving partial squares}\\ &\le (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon.\end{aligned}$$

Since $4\sqrt{2}S^2 + \sqrt{2}SL$ is constant for given C and $\varepsilon > 0$ can be made arbitrarily small, then we must have $\int_C f(z)\,dz = 0$, as claimed. The proof for C negatively oriented is similar (requiring similarly modified lemmas). □

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