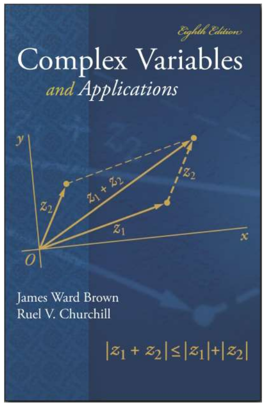
Complex Variables

Chapter 4. Integrals

Section 4.48. Simply Connected Domains—Proofs of Theorems



Theorem 4.48.A

Theorem 4.48.A. If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)\,dz = 0$ for every closed contour C lying in D.

Proof for Some Closed Contours. If C is a simple closed contour then the claim holds by the Cauchy-Goursat Theorem (Theorem 4.46.A) since the points interior to C are all in D.

If C is closed but intersects itself a finite number of times, it can be partitioned into a finite number of simple closed contours, $C = C_1 \cup C_2 \cup \cdots \cup C_n = C_1 + C_2 + \cdots + C_n$ (see Figure 58 for an example where $n = 4$).

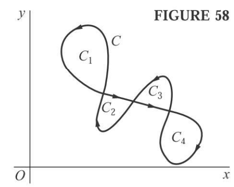


FIGURE 58

Theorem 4.48.A (continued)

Theorem 4.48.A. If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)\,dz = 0$ for every closed contour C lying in D.

Proof for Some Closed Contours (continued). Then

$$\int_C f(z)\,dz = \int_{\cup_{k=1}^n C_k} f(z)\,dz = \sum_{k=1}^n \left(\int_{C_k} f(z)\,dz \right) = 0$$

(see section 4.40), by the first part of this proof. □

Corollary 4.48.B

Corollary 4.48.B. A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Proof. Let C be a closed contour in D. Then by Theorem 4.48.A, $\int_C f(z)\,dz = 0$. Since f is analytic throughout D then f is continuous on D. So by Theorem 4.44.A (the (c) implies (a) part), f has an antiderivative throughout D. □