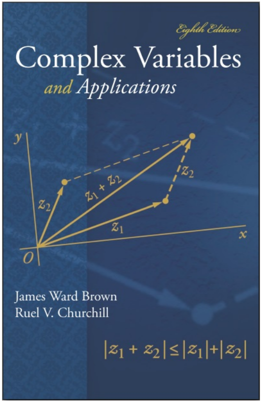
Complex Variables

Chapter 4. Integrals

Section 4.48. Simply Connected Domains—Proofs of Theorems



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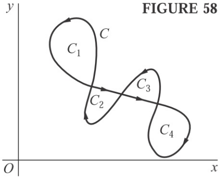

FIGURE 58

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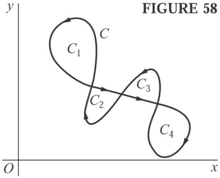

FIGURE 58

Theorem 4.48.A (continued)

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$$\int_C f(z)\,dz = \int_{\cup_{k=1}^n C_k} f(z)\,dz = \sum_{k=1}^n \left(\int_{C_k} f(z)\,dz \right) = 0$$

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Corollary 4.48.B

Corollary 4.48.B. A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Proof. Let C be a closed contour in D. Then by Theorem 4.48.A, $\int_C f(z)\,dz = 0$. Since f is analytic throughout D then f is continuous on D. So by Theorem 4.44.A (the (c) implies (a) part), f has an antiderivative throughout D. □

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