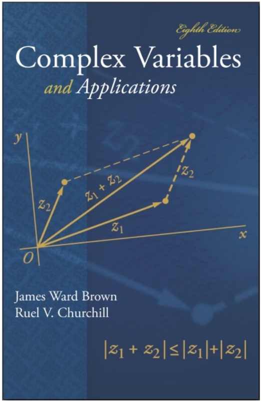
Complex Variables

Chapter 4. Integrals

Section 4.49. Multiply Connected Domains—Proofs of Theorems



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Theorem 4.49.A

Theorem 4.49.A. Suppose that

(a) C is a simple closed contour, parameterized in the counterclockwise direction; and

(b) C_k, for $k = 1, 2, \ldots, n$, are simple closed contours interior to C, all parameterized in the clockwise direction, that are disjoint and whose interiors have no points in common (so one such contour cannot be inside another).

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

"Proof." We assume some obvious properties of simple closed contours and assume that we can introduce a polygonal path L_1 consisting of a finite number of line segments joined end to end to connect the outer contour C to the inner contour C_1.

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Theorem 4.49.A (continued 1)

"Proof" continued. Introduce polygonal path L_2 connecting C_1 to C_2, and in general polygonal path L_{k+1} connecting C_k to C_{k+1} for $k = 1, 2, \ldots, n-1$. Introduce polygonal path L_{n+1} connecting C_n to outer contour C. See Figure 60 (In Figure 60, each polygonal path is a single line segment and the L_k are disjoint; these need not be the case).

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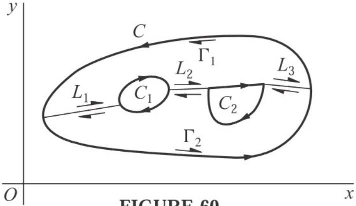

FIGURE 60

Theorem 4.49.A (continued 1)

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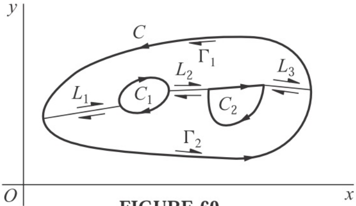

FIGURE 60

As shown in Figure 60, two simple closed contours Γ_1 and Γ_2 can be formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them "lie to the left." (This, granted, is an informal "proof by picture.")

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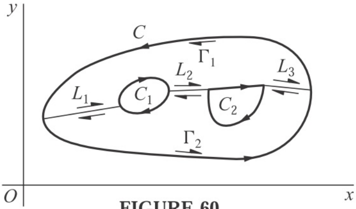

FIGURE 60

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Theorem 4.49.A (continued 2)

"Proof" continued. By the Cauchy-Goursat Theorem (Theorem 4.46.A), $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz = 0$. So (see Section 40)

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0 = \int_{\Gamma_1 \cup \Gamma_2} f(z)\,dz.$$

Now for each $k = 1, 2, \ldots, n+1$, the integral $\int_{L_k} f(z)\,dz$ and $\int_{-L_k} f(z)\,dz$ are parts of $\int_{\Gamma_1 \cup \Gamma_2} f(z)\,dz$, but $\int_{-L_k} f(z)\,dz = -\int_{L_k} f(z)\,dz$ (see Note 4.40.B) so these cancel each other and

$$0 = \int_{\Gamma_1 \cup \Gamma_2} f(z)\,dz = \int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz,$$

as claimed. □

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as claimed. □

Corollary 4.49.B

Corollary 4.49.B. Principle of Deformation.
Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Proof. By Theorem 4.49.A,

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

(we need to use $-C_1$ to get the inner contour parameterized in a clockwise direction; see Figure 61).

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Proof (continued).

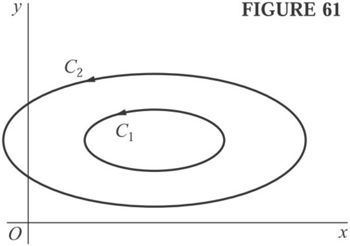

FIGURE 61

So

$$\int_{C_2} f(z)\,dz = -\int_{-C_1} f(z)\,dz = \int_{C_1} f(z)\,dz$$

(see Note 4.40.B). □